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MATRIX ELEMENTS OF THE NUCLEAR INERTIAL AND GYROMAGNETIC TENSOR BETWEEN VIBRATIONAL STATES OF EVEN RARE EARTH NUCLEI

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Abstract

The effective Coriolis coupling matrix elements between the intrinsic ground state and β- and γ-vibrational states are calculated from a non-adiabatic, microscopic model. Within the same model we consider the effective gyromagnetic tensor. The matrix elements obtained are applied in a calculation of M1 and E2 transition probabilities. A comparison is made with experimental data. The results are compared also with those obtained in an adiabatic approximation, and it is shown that this is a good approximation only for the matrix elements of the inertial and gyromagnetic tensor between the ground state and a vibrational state.

1. Introduction

In recent years there has been a strong interest in the study of branching ratios of the γ -transitions from the lowest, excited rotational bands of even, deformed nuclei. Accurate measurements of the relative intensities of an increasing number of transitions have shown[35] that in general a simple parametrization[1] in terms of first order corrections due to the rotation-vibration interaction between the ground state band and the excited band does not account for the observed deviations from the Alaga-rules. Accordingly various versions of a more complicated analysis, involving mostly the assumption of an additional Coriolis coupling between the β - and γ - vibrational band, have been applied by a number of authors[2-5].

In general this type of phenomenological analysis is based on the assumption of a Hamiltonian

$$H_{total} = H_{intr} + h_0 (I(I+1) - K^2) + h_2 I_-^2 + h_{-2} I_+^2 . \quad (1)$$

Here we have used the conventional notation[6]. The coefficients h_0 and $h_{\pm 2}$ are operators acting on the intrinsic wave function. In the present work we shall be concerned with the calculation of the matrix elements of these operators from a microscopic model. Obviously the operators h_0 and $h_{\pm 2}$ may be viewed as components of a half inverse inertial tensor of the nucleus. We shall consider also the matrix elements of

the closely related gyromagnetic tensor, which is connected from an experimental point of view to magnetic moments and M1 transition probabilities.

Microscopic models for the Coriolis coupling between the even-K vibrational states have been considered previously by Pavlichenkov[7], Bès et al.[8], Marshalek[9], and Kumar and Baranger[10]. In the works of Marshalek, and Kumar and Baranger, the assumption of an adiabatic vibrational motion was utilized. Marshalek treated only the coupling between the intrinsic ground state and a vibrational state. In the approach of Kumar and Baranger the Coriolis coupling is implicit in a numerical treatment of the total Bohr Hamiltonian[15,25].

One important conclusion of the present work is the necessity of a non-adiabatic treatment of the vibrational motion, if one wants to calculate the matrix elements of the inertial and gyromagnetic tensor between the excited vibrational states.

Pavlichenkov and Bès et al. both use a non-adiabatic formalism. In the work of the latter authors the rotation-vibration interaction is considered, however, only in so far as it influences the electromagnetic transition probabilities between the γ -band and the ground state band.

Pavlichenkov's work is that previous investigation, which is most closely related to the present one. This author considers the matrix elements of the inertial tensor between the ground state and the vibrational states as well as the diagonal matrix elements between these states. The basic equations used are very similar to those obtained in the present work, though

the method is somewhat different. However the numerical calculations made by Pavlichenkov are restricted to the unrealistic case of the pure harmonic oscillator single particle model.

In the present work all five independent matrix elements of the inertial tensor between the following intrinsic states are calculated from a non-adiabatic, microscopic model: $|0\rangle$ ≡ ground state, $|\beta\rangle$ ≡ single phonon β -vibrational state†, and $|\gamma\rangle$ and $|\bar{\gamma}\rangle$ ≡ single phonon γ -vibrational state with, respectively, $K = \pm 2$.

Footnote

†By definition the state $|\beta\rangle$, $|\gamma\rangle$, or $|\bar{\gamma}\rangle$ is taken to be the lowest excited state with the appropriate quantum numbers K^{π}, independent of the degree of collectivity of this state.

(The ground state moment of inertia is treated as an empirical parameter.) Thus for the first time a non-aidabatic microscopic calculation is made of the non-diagonal matrix element of the inertial tensor between the states $|\beta\rangle$ and $|\gamma\rangle$. The matrix elements of the gyromagnetic tensor between the same four states are also calculated.

The way this is done is simple in its principle. The basic expression for the inertial tensor is just a generalization of the cranking formula[29] to a matrix form corresponding to the multi-dimensional intrinsic configuration space considered. (Cf. eqs. (23) and (27).) The foundation of this matrix generalization of the cranking formula is established in

sects. 2 and 3. In sect. 4 we consider its implications for the matrix elements of electromagnetic moments and derive the expression for the gyromagnetic tensor.

The theory of sects. 2-4 is general. It applies in principle to any deformed model Hamiltonian chosen to represent the energy of the nucleus in a configuration with a definite orientation in space. In sect. 5 a definite model Hamiltonian is chosen, namely, effectively, the Hamiltonian of the pairing plus quadrupole model with a slight modification that will be explained. In sect. 6 we derive the relevant approximative expressions for the quantities considered from a perturbation expansion in the residual interaction.

Details of our calculation are explained in sect. 7. In sect. 8 we present our results and make a comparison with empirical data. We calculate the matrix elements of $\underline{b}_0$ and $\underline{b}_{+2}$, the values of B(M1, $2 \rightarrow 2'$) between the three bands, and the values of B(E2, $0_0 \rightarrow 2_\beta$ or 2_γ). The comparison with an adiabatic calculation is also made in sect. 8.

In sect. 9 we draw the conclusion from our work. The details of our graphical formulation of the perturbation expansion in the residual interaction are given in four appendices.

2. Cranking formalism with intrinsic degrees of freedom

From the presence of regular rotational bands we expect that the rotational part of the nuclear motion is fairly well described in the cranking model approximation. We shall apply in the present work a cranking formalism for a system with intrinsic degrees of freedom, which is based on a generalization of the variational approach[6] to the cranking model for a single

rotational band. The advantage of this foundation of our theory will be explained later. (Cf. the concluding remark of sect. 3.) In the present section we set up the basic equations of the formalism.

As usual for the nuclei considered we assume that the rotating system has the intrinsic symmetry associated with the subgroup[11] $\mathcal{D}_\infty$ of the group of rotations. Thus only the two degrees of freedom corresponding to a variation of a unit vector along the symmetry axis on a hemisphere are rotational ones.

According to the cranking assumption we represent the exact rotational states by rotating wave packets in the form of many-body wave functions with a certain average orientation and average angular momentum. Let a set $\bar{\Psi}_0, \ldots, \bar{\Psi}_{\underline{d}}$ of such wave packets represent $\underline{d}+1$ intrinsic states of a nucleus, which is oriented along the 3-axis of a given Cartesian coordinate system and has an angular momentum $\underline{I}$ with the components $\underline{I}_1$ and $\underline{I}_2$ in this coordinate system. The orientation and angular momentum are defined classically as averages over the given wave packet. The intrinsic variables, which distinguish the different states $\bar{\Psi}_{\underline{n}}$, must commute by definition with the unit operator and the components $\underline{J}_1$ and $\underline{J}_2$ of the quantal angular momentum $\underline{J}$. Thus, assuming that the wave functions are normalized, we have

$$\bar{\Psi}_n^\dagger \bar{\Psi}_m = \delta_{nm}, \qquad (2)$$

= erpen- icular

$$\underline{\Psi}_n^{\dagger}\, \underline{\underline{J}}_{\perp}\, \underline{\Psi}_m = \underline{\underline{I}}_{\perp}\, \delta_{nm} \,, \tag{3}$$

where $\underline{\underline{J}}_{\perp} \equiv (\underline{J}_1, \underline{J}_2)$, $\underline{\underline{I}}_{\perp} \equiv (\underline{I}_1, \underline{I}_2)$. We note that the eqs. (2) and (3) are invariant under arbitrary unitary transformations among the wave functions $\underline{\Psi}_n$. Hence they describe in reality a property of the $(\underline{d}+1)$-dimensional space spanned by these wave functions.

In order to describe the lowest rotational bands in the nuclear spectrum we select that space with the properties (2) and (3), which has the smallest average expectation energy

$$\langle E \rangle \equiv \frac{1}{d+1} \sum_{n=0}^{d} \underline{\Psi}_n^{+}\, H\, \underline{\Psi}_n \,. \tag{4}$$

We assume here that $\underline{H}$ is a deformed model Hamiltonian with D_∞-symmetry in the coordinate system considered. Hereby the condition of minimal $\langle \underline{E} \rangle$ defines automatically a space of wave packets with orientation along the 3-axis, so that no additional constraints are needed to ensure the proper orientation.

The wave functions $\underline{\Psi}_n$ are determined thus as functions of $\underline{\underline{I}}_{\perp}$ within an arbitrary, $\underline{\underline{I}}_{\perp}$-dependent, unitary transformation among them. These transformations will correspond in the final formalism to canonical transformations, which commute with the components of the angular momentum in the laboratory system. A linear combination

$$\underline{\Psi} = \sum_{n=0}^{d} c_n\, \underline{\Psi}_n(\underline{\underline{I}}_{\perp}) \tag{5}$$

has the expectation energy

$$\underline{\Psi}^{+} H\, \underline{\Psi} = \sum_{n,m=0}^{d} c_n^{*} c_m\, H_{nm}(\underline{\underline{I}}_{\perp}) \tag{6}$$

with

$$H_{nm}(\underline{\underline{I}}_{\perp}) \equiv \underline{\Psi}_n(\underline{\underline{I}}_{\perp})^{+}\, H\, \underline{\Psi}_m(\underline{\underline{I}}_{\perp}). \tag{7}$$

Hence the matrix $\underline{H}_{nm}(\underline{\underline{I}}_{\perp})$ may be taken as an effective Hamiltonian for the model space of lowest rotational bands considered. This Hamiltonian is a function of the classical angular momentum components $\underline{I}_1$ and $\underline{I}_2$. A quantized model is obtained by replacing these variables by the corresponding differential operators in the expression for a given matrix element $\underline{H}_{nm}(\underline{\underline{I}}_{\perp})$ and letting the resulting operator matrix act on a vector $\underline{\chi}_n(\varphi, \vartheta, \psi)$, which is a function of the Euler angles of the coordinate system. If the minimum of $\langle \underline{E} \rangle$ is unique - which requires a suitable value of $\underline{d}$ - then the freedom in the choice of the wave functions $\underline{\Psi}_n(\underline{\underline{I}}_{\perp})$ already mentioned can be utilized to provide a representation of the group D_∞ by matrices acting on the vector $\underline{\chi}_n$, in terms of which the constraints on $\underline{\chi}_n(\underline{\varphi}, \underline{\vartheta}, \underline{\psi})$ associated with an intrinsic D_∞-symmetry of the rotating system may be imposed in the usual manner[6].

More generally we have an effective matrix

$$A_{nm}(\underline{I}_\perp,\varphi,\vartheta,\psi) \equiv \underline{\Psi}_n(\underline{I}_\perp)^+ A(\varphi,\vartheta,\psi)\,\underline{\Psi}_m(\underline{I}_\perp) \qquad (8)$$

associated with any many-body operator $\underline{A}(\varphi,\vartheta,\psi)$, which is a D_∞ -invariant function of the Euler angles. In sect. 4 we shall apply eq. (8) to M1 and E2 moment operators.

As in the case of a single intrinsic state we solve the variational problem

$$\underline{\delta}\langle E\rangle = 0 \qquad (9)$$

under the constraints (2) and (3) by means of Lagrangian multipliers. Hereby we obtain a set of coupled equations

$$H\underline{\Psi}_n = \sum_{m=0}^{d} \left(\underline{J}_\perp \underline{\Psi}_m \cdot\cdot \underline{\omega}_{mn} + \underline{\Psi}_m \underline{E}'_{mn} \right), \qquad (10)$$

where $\underline{\omega}_{mn} \equiv (\omega^1_{mn}, \omega^2_{mn})$ and ω^1_{mn}, ω^2_{mn}, and $\underline{E}'_{mn}$ are Hermitean matrices. When the matrix elements $\underline{\omega}_{mn}$ are sufficiently small, we can make a perturbation expansion in these quantities. We shall use an orthonormalized, multi-dimensional Brioullin-Wigner scheme[12]. Thus let $|n\rangle$, $|m\rangle$, $|p\rangle$, and $|q\rangle$ be eigenstates of H, chosen among the $d+1$ lowest ones, let $|i\rangle$ denote an eigenstate outside this set, and let $E_n, E_m, \ldots$ be the energies of the states $|n\rangle, |m\rangle, \ldots$. We assume that all eigenfunctions are chosen with the conventional phase, so that $\exp(-i\pi J_2)|n\rangle = |\bar{n}\rangle$, where the bar denotes

time-reversal conjugation. Projecting eq. (10) on a wave function $|i\rangle$ we obtain, to first order in $\underline{\omega}_{nm}$,

$$\langle i|\underline{\Psi}_n\rangle = \sum_m \frac{\langle i|\underline{J}_\perp|m\rangle \cdot\cdot \underline{\omega}_{mn}}{E_i - E_n} \,. \qquad (11)$$

Hereby eq. (2) is satisfied to second order by the choice

$$\langle m|\underline{\Psi}_n\rangle = \delta_{mn} - \frac{1}{2}\sum_{pqi} \frac{\underline{\omega}_{mp} \cdot\cdot \langle p|\underline{J}_\perp|i\rangle\langle i|\underline{J}_\perp|q\rangle \cdot\cdot \underline{\omega}_{qn}}{(E_i - E_p)(E_i - E_q)} \,. \qquad (12)$$

When the wave functions $\underline{\Psi}_n$ are given, the matrix $\underline{E}'_{nm}$ is uniquely determined, e.g. by the equations obtained by projecting eq. (10) on the wave functions $|m\rangle$, and the constraints (2) and (3) in connection with eq. (10) ensure the Hermitecity of $\underline{E}'_{nm}$. Thus eqs. (3), (11), and (12) determine an approximate solution of the variational problem. Note that the solution (12) is ambiguous within a left hand multiplication by an arbitrary unitary matrix. This ambiguity reflects the invariance of the basic equations (2), (3), and (9) mentioned above. The Hermitean expression (12) corresponds in the order considered to the choice of the so-called 'midway-basis' in the perturbed space[47].

When the left hand side of eq. (3) is expanded to first order in $\underline{\omega}_{nm}$ by means of eqs. (11) and (12) we obtain a set of linear eqations, which can be written in the form

Script capital $\mathcal{J}$

$$\underline{\omega}^{\alpha}_{nm} = \frac{1}{\mathcal{J}}\Big(\delta_{nm} I_{\alpha} - \langle n|\mathcal{J}_{\alpha}|m\rangle - \sum_{p\beta}\Big(\langle n|\delta\mathcal{J}^{m}_{\alpha\beta}|p\rangle \underline{\omega}^{\beta}_{pm} + \underline{\omega}^{\beta}_{np}\langle p|\delta\mathcal{J}^{n}_{\beta\alpha}|m\rangle\Big)\Big) \tag{13}$$

with $\alpha,\beta = 1$ or 2. Here

$$\mathcal{J} \equiv 2\sum_{i}\frac{\langle o|\mathcal{J}_{\perp}|i\rangle^{2}}{E_i - E_o}, \tag{14}$$

where $|o\rangle$ is the ground state of H, and

$$\langle n|\delta\mathcal{J}^{p}_{\alpha\beta}|m\rangle \equiv \sum_{i}\frac{\langle n|\mathcal{J}_{\alpha}|i\rangle\langle i|\mathcal{J}_{\beta}|m\rangle}{E_i - E_p} - \frac{1}{2}\delta_{nm}\delta_{\alpha\beta}\mathcal{J}. \tag{15}$$

Note that $\langle o|\delta\mathcal{J}^{o}_{\alpha\beta}|o\rangle = 0$. From Eqs. (7), (8), (11), (12), (14), and (15), we get, respectively to second and first order in $\underline{\omega}_{nm}$,

$$H_{nm}(\underline{I}_{\perp}) = \delta_{nm}E_m + \frac{1}{2}\Big(\mathcal{J}\sum_{p}\underline{\omega}_{np}\cdot\underline{\omega}_{pm} + \sum_{pq\alpha\beta}\underline{\omega}^{\alpha}_{np}\langle p|(\delta\mathcal{J}^{m}_{\alpha\beta} + \delta\mathcal{J}^{n}_{\alpha\beta})|q\rangle\underline{\omega}^{\beta}_{qm}\Big), \tag{16}$$

$$A_{nm}(\underline{I}_{\perp}) = \langle n|A|m\rangle + \sum_{pi}\left(\frac{\langle n|A|i\rangle\langle i|\mathcal{J}_{\perp}|p\rangle\cdot\underline{\omega}_{pm}}{E_i - E_m} + \frac{\underline{\omega}_{np}\cdot\langle p|\mathcal{J}_{\perp}|i\rangle\langle i|A|m\rangle}{E_i - E_n}\right). \tag{17}$$

It is understood here that $\underline{\omega}_{nm}$ is a function of $\underline{I}_{\perp}$ through eq. (13), and that $\underline{A}$ may depend on the Euler angles.

Eqs. (13)-(17) are the basic equations that will be used in the following.

The quantization of the classical equations leaves the order of the factors in products of the operators $\underline{I}_1$, $\underline{I}_2$, φ, ϑ, and ψ undetermined to a certain extend. A Hermitean representation of Hermitean many-body operators is obtained, if we write, for example, the operators $\underline{I}_1, \underline{I}_2, \varphi, \vartheta$, and ψ in exactly that order, in which they appear in the eqs. (16) and (17), when the solution of eq. (13) is inserted. We shall assume in the following that this is the proper recipe. It will turn out that it leads to a formal equivalence between a certain approximation of our theory and a perturbation expansion in the Coriolis interaction as it is made e.g. in the work of Près et al.[8]. (Sect. 4.)

3. The inertial tensor

In order to apply the formalism of sect. 1 in our actual case we shall now suppose that the basic equations (13)-(17) are applicable in a slightly generalized situation. Thus we assume that all the states $|n\rangle$ are even-K states, and that d is chosen as a large number. (Formally we just include all even-K states.) Making the perturbation expansion from these states we actually obtain an extremum of $\langle E\rangle$ rather than a minimum. The present generalization corresponds closely to the generalization, which is involved when the usual cranking formula is applied to excited bands.

The set of all even-K states contains the states $|0\rangle$, $|\beta\rangle$, $|\gamma\rangle$, and $|\bar{\gamma}\rangle$. For brevity we shall name these states the states υ.

It will be convenient to use the complex representation of vectors in the 1-2-plane. Thus we consider vector components like $I_\pm = I_1 \pm i I_2$. As the metric tensor of the complex representation, $G^{\alpha\beta} = \frac{1}{2}\delta_{\alpha(-\beta)}$, is different from the unit matrix, it is adequate to distinguish between contravariant components, like $I_\pm$, and covariant components $I^\alpha \equiv \sum_\beta G^{\alpha\beta} I_\beta$, i.e. $I^\pm = \frac{1}{2}(I_1 \mp i I_2)$. In sect. 1 we have anticipated this notation. Thus eqs. (14)-(17) are written already in an invariant form. In eq. (13) a matrix multiplication by $G^{\alpha\beta}$ must be inserted on the right hand side.

As all odd-K states appear among the states $|i\rangle$, eq. (14) becomes the normal cranking formula for the ground state moment of inertia. When the expression for the model Hamiltonian H includes a residual interaction, as we shall assume below (sect. 5), $\mathcal{J}$ contains the corresponding contributions, which are accounted for by the Thouless-Valatin[48] expression. (Cf. theorem 1 in appendix B. The Thouless-Valatin formula corresponds to a summation of all 'RPA-type' diagrams. This gives exactly the expression in eq. (14), when the effective RPA-interaction (appendix C) for the frequency zero is inserted on the place of an interaction line.)

From eqs. (13) and (16) we get an expression for the effective total Hamiltonian of the form (1), namely

$$H_{nm}(I_\perp) \equiv \delta_{nm} E_m + \sum_{\alpha\beta} \langle n | h_a^{\alpha\beta} | m \rangle I_\alpha I_\beta$$

$$= \delta_{nm} E_m - \frac{i}{2} \langle n | h_a^{+-} - h_a^{-+} | m \rangle + \sum_{\alpha\beta} \langle n | h^{\alpha\beta} | m \rangle I_\alpha I_\beta \quad (19)$$

with

$$h^{\alpha\beta} \equiv \frac{1}{2}\left(h_a^{\alpha\beta} + h_a^{\beta\alpha}\right). \quad (20)$$

The operators h_0 and $h_{\pm 2}$ are proportional to the components of the tensor $h^{\alpha\beta}$. Thus

$$h_0 = 2h^{+-} = 2h^{-+}, \quad (21a)$$

$$h_{\pm 2} = h^{\mp\mp}. \quad (21b)$$

We can neglect the second term in the third member of eq. (19), as these matrix elements are very small compared to the typical separation of the energies E_n. If we let $h^{(0)\alpha\beta}, h^{(1)\alpha\beta}, h^{(2)\alpha\beta}, \ldots$ denote the contributions to $h^{\alpha\beta}$ obtained in successive orders of an expansion in the matrix elements of $\delta\mathcal{J}^n_{\alpha\beta}$, we get after some manipulations,

$$h^{(0)\alpha\beta} = \frac{1}{2\mathcal{J}} G^{\alpha\beta}, \quad (22a)$$

$$h^{(1)\alpha\beta} = -\frac{1}{4\mathcal{J}^2}\left(\delta\mathcal{J}^{\alpha\beta} + \delta\mathcal{J}^{\beta\alpha}\right), \quad (22b)$$

$$\langle n | h^{(2)\alpha\beta} | m \rangle = \frac{1}{4\mathcal{J}^3} \sum_\gamma \langle n | \left(\delta\mathcal{J}_\gamma{}^\alpha \delta\mathcal{J}^{\gamma\beta} + \delta\mathcal{J}_{\bar{\gamma}}{}^\beta \delta\mathcal{J}^{\bar{\gamma}\alpha}\right.$$

$$+ \left(\delta\mathcal{J}^{m\alpha}_\gamma - \delta\mathcal{J}^{n\alpha}_{\bar{\gamma}}\right)\delta\mathcal{J}^{\gamma\beta} + \left(\delta\mathcal{J}^{m\beta}_\gamma - \delta\mathcal{J}^{n\beta}_{\bar{\gamma}}\right)\delta\mathcal{J}^{\gamma\alpha}$$

$$\left. - \delta\mathcal{J}^\alpha_\gamma\left(\delta\mathcal{J}^{m\gamma\beta} - \delta\mathcal{J}^{n\bar{\gamma}\beta}\right) - \delta\mathcal{J}^\beta_\gamma\left(\delta\mathcal{J}^{m\gamma\alpha} - \delta\mathcal{J}^{n\bar{\gamma}\alpha}\right)\right) | m \rangle, \quad (22c)$$

where products like $\delta\mathcal{J}_\gamma^{\alpha}\delta\mathcal{J}^{\gamma\beta}$ should be understood as operator products in the space of even-K states, and

$$\langle n|\delta\mathcal{J}_{\alpha\beta}|m\rangle \equiv \langle n|(\delta\mathcal{J}^m_{\alpha\beta}+\delta\mathcal{J}^n_{\beta\alpha})|m\rangle$$
$$=\sum_i\left(\frac{\langle n|J_\alpha|i\rangle\langle i|J_\beta|m\rangle}{E_i-E_m}+\frac{\langle n|J_\beta|i\rangle\langle i|J_\alpha|m\rangle}{E_i-E_n}\right). \qquad (23)$$
$$-\delta_{nm}G_{\alpha\beta}\mathcal{J}$$

Note that $\langle n|\delta\mathcal{J}_{+-}|m\rangle=\langle n|\delta\mathcal{J}_{-+}|m\rangle$, when $E_n=E_m$ or $K_n=K_m=0$. Hence, if only matrix elements between the states $\mathcal{V}$ are considered, we can write

$$h^{(1)\alpha\beta}=-\frac{1}{2\mathcal{J}^2}\delta\mathcal{J}^{\alpha\beta}=-\frac{1}{2\mathcal{J}^2}\delta\mathcal{J}^{\beta\alpha}. \qquad (22b')$$

The four terms in eq. (22c), which contain differences of the form $\delta\mathcal{J}^m_{\alpha\beta}-\delta\mathcal{J}^n_{\alpha\beta}$ tend to cancel each other as seen by writing down the explicit expressions by means of eqs. (15) and (23). In addition each of them vanishes, if $E_n=E_m$. As a probably good approximation we can therefore neglect these contributions to the matrix elements between the states $\mathcal{V}$, which are fairly degenerate compared to an effective energy of the odd-K excitations in the expression (15). Thus we write

$$h^{(2)\alpha\beta}=\frac{1}{4\mathcal{J}^3}\sum_\gamma\left(\delta\mathcal{J}_\gamma^{\alpha}\delta\mathcal{J}^{\gamma\beta}+\delta\mathcal{J}_\gamma^{\beta}\delta\mathcal{J}^{\gamma\alpha}\right). \qquad (22c')$$

In sect. 6 we shall derive explicit expressions for the matrix elements of $\delta\mathcal{J}_{\alpha\beta}$ between the states $\mathcal{V}$ from a perturbation expansion of the pairing plus quadrupole model Hamiltonian. From these matrix elements the corresponding matrix

elements of $h^{(1)\alpha\beta}$ are obtained immediately by means of eq. (22b'). In order to calculate the matrix elements of $h^{(2)\alpha\beta}$ we must, however, according to eq. (22c') know in principle all matrix elements of $\delta\mathcal{J}_{\alpha\beta}$ between the states $\mathcal{V}$ and an arbitrary even-K state. To get a practically computable expression we assume that the unknown matrix elements between the states $\mathcal{V}$ and other even-K states can be simulated by writing $\delta\mathcal{J}_{\alpha\beta}$ in the form

$$\delta\mathcal{J}_{\alpha\beta}=\sum_\mu\langle o|\delta\mathcal{J}_{\alpha\beta}|\mu\rangle\left(B_\mu+B^\dagger_{-\mu}\right)$$
$$+\sum_{\mu\nu}\langle\mu|\delta\mathcal{J}_{\alpha\beta}|\nu\rangle B^\dagger_\mu B_\nu, \qquad (24)$$

† = Hermitean conjugate

where $|\mu\rangle$ denotes the single phonon state in $\mathcal{V}$ with $K=\mu$, and B_μ the annihilation operator of a corresponding harmonic oscillator quantum. Obviously the expression (24) has the known matrix elements between the states $\mathcal{V}$. Using the eqs. (22c') and (24) it is trivial to derive expressions for the matrix elements of $h^{(2)\alpha\beta}$ between these states. Note especially that the $h^{(2)\alpha\beta}$-term gives a contribution to the effective ground state moment of inertia $\mathcal{J}_{eff}$ From the eqs. above we get

$$\frac{1}{\mathcal{J}_{eff}}\equiv 2\langle o|h_o|o\rangle=\frac{1}{\mathcal{J}}+\frac{1}{8\mathcal{J}^3}\sum_{\alpha\beta\mu}\langle o|\delta\mathcal{J}_{\alpha\beta}|\mu\rangle^2$$
$$=\frac{1}{\mathcal{J}}+4\mathcal{J}\left(\langle o|h^{(1)}_0|\beta\rangle^2+4\langle o|h^{(1)}_{-2}|\gamma\rangle^2\right). \qquad (25)$$

We note that with the assumption (24) the relation $\delta\mathcal{J}_{+-}=\delta\mathcal{J}_{-+}$ applies generally. In this case the eqs. (22a), (22b'), and (22c') correspond simply to a Taylor expansion of the inverse tensor in the expression

$$h^{\alpha\beta} = \frac{1}{4}\left((\mathcal{J}^{-1})^{\alpha\beta} + (\mathcal{J}^{-1})^{\beta\alpha}\right), \qquad (26)$$

where (cf. eq. (23))

$$\mathcal{J}_{\alpha\beta} \equiv \mathcal{J}\,G_{\alpha\beta} + \delta\mathcal{J}_{\alpha\beta}, \qquad (27)$$

and the inverse tensor is defined by

$$\sum_{\gamma}(\mathcal{J}^{-1})^{\alpha\gamma}\mathcal{J}_{\gamma\beta} = \sum_{\gamma}\mathcal{J}_{\alpha\gamma}(\mathcal{J}^{-1})^{\gamma\beta} = \delta_{\alpha\beta}. \qquad (28)$$

Thus it is natural to interpret the operators $\mathcal{J}_{\alpha\beta}$ as components of a quantum mechanical inertial tensor. (The symmetry of $\mathcal{J}_{\alpha\beta}$ implies a symmetry of the inverse tensor, only when the different tensor components commute. This must be satisfied in the adiabatic limit (see below), where the inertial tensor is a function of the deformation coordinates alone, but it need not be satisfied away from this limit.)

With respect to the assumption of a quadratic boson expansion of the inertial tensor our approach is similar to that of Pavlichenkov[7)†]. In Pavlichenkov's work the inertial tensor is constructed by means of a Bogoliubov transformation, which minimizes the expectation value of the operator $H - \omega_x J_x$ in the quasiparticle vacuum. Here the angular velocity ω is treated as a classical vector. The transformed operator H' contains a term of the form $-\sum_{\alpha\beta}\mathcal{J}_{\alpha\beta}\omega^{\alpha}\omega^{\beta}$ where the coefficients $\mathcal{J}_{\alpha\beta}$ are operators. The operators $\mathcal{J}_{+-}$ and $\mathcal{J}_{++}$ do not commute in this expression. By means of a usual RPA-linearization Pavlichen-

Footnote

†Note, however, that we do not assume that the two-phonon states are stationary states of the system.

kov obtains a boson expansion of $\mathcal{J}_{\alpha\beta}$, where - as we have done it here - he neglects the terms associated with non-collective bosons. In ref.[7)] terms of the type $B^{\dagger}_{\mu}B_{\nu}$ with $\mu \neq \nu$ are also neglected. The coeficients of $B^{\dagger}_{\mu}B_{\mu}$ are not evaluated directly from the Bogoliubov transformation and RPA-linearization. Instead a variation of the Bethe-Salpeter equation is considered. From the classical relation $I_{\alpha} = -\partial H'/\partial\omega^{\alpha}$ Pavlichenkov obtains an expansion of $h^{\alpha\beta}$ similar to our expansion $h^{\alpha\beta} = h^{(0)\alpha\beta} + h^{(1)\alpha\beta} + h^{(2)\alpha\beta}$. In the term $h^{(2)\alpha\beta}$ he takes into account only the linear part of the expression (24). Hereby he avoids the commutability problem. We shall see, however, that sometimes the matrix elements $\langle\mu|\delta\mathcal{J}_{\alpha\beta}|\nu\rangle$ are comparable with the matrix elements $\langle 0|\delta\mathcal{J}_{\alpha\beta}|\mu\rangle$. It is therefore reasonable to include the products involving the quadratic term in eq. (24) in the expression for $h^{(2)\alpha\beta}$. In the corresponding order of $\delta\mathcal{J}_{\alpha\beta}$ the solution of the classical equation then becomes ambiguous due to the non-commutability of the tensor components $\mathcal{J}_{\alpha\beta}$, and some additional assumption, like e.g. the intuitive eq. (26), is needed.

Contrary to the expression (24) the boson expansion written in ref. [7)] contains also terms proportional to $B^{\dagger}_{\mu}B^{\dagger}_{-\mu} + B_{-\mu}B_{\mu}$. However their coeficients are not evaluated there. An attempt made in connection with the present work to evaluate these coeficients assuming harmonicity of the vibrations and including only the leading term in a perturbation expansion in the residual interaction (cf. sect. 6) turned out to give unphysical results. A reliable estimate requires that anharmonicities of the vibrational modes are taken into account. Therefore we have not included such terms in eq. (24). One cannot, however, rule out the possibility that these terms may contribute significantly to the matrix elements $\langle\mu|h^{\alpha\beta}|\nu\rangle$. Thus in the adiabatic limit the terms $B^{\dagger}_{\mu}B_{\nu}$ and $\frac{1}{2}(B^{\dagger}_{\mu}B^{\dagger}_{\nu} + B_{\nu}B_{\mu})$

must have equal coeficients.

In Pavlichenkovs construction a special significance is assigned to a single intrinsic state, the vacuum of Bogoliubov quasiparticles. Accordingly the fundamental step, the solution of the linear equation $\underline{I}_\alpha = -\partial \underline{H}'/\partial \omega^\alpha$ becomes generally ambigous, because the Lagrangean multipliers ω^α, which are introduced first as classical quantities, must be reinterpreted later effectively as operators acting on the intrinsic part of the rotor model wave function. With the present construction all intrinsic states considered are treated on an equal footing. Accordingly the Lagrangean multipliers ω^α are matrices (operators) from the beginning, and the ambiguity just mentioned is avoided. The cost is that the method must be based on the generalized variational principle (9), the justification of which is not clear. Yet our approach is quasi-classical. The - less serious - ambiguity associated with the non-commutability of the operators $\underline{I}_1, \underline{I}_2, \varphi, \vartheta,$ and ψ persists.

It turns out that with our generalized treatment of the Lagrangean multipliers $\underline{\omega}^\alpha$ the intuitive equation (26) is only approximately true. A more careful analysis of the general significance of the neglected terms in eq. (22c) remains to be carried out.

4. Matrix elements of electromagnetic moment operators

In the present formalism two sources of deviations from the Alaga-rules are distinguished, namely: 1) 'Band mixing' of the even-$\underline{K}$ bands associated with off-diagonal elements of the matrix $\underline{H}_{nn'}(\underline{I}_\perp)$. 2) Mixing with the odd-$\underline{K}$ bands resulting in the renormalization of the 'intrinsic' transition matrix elements, given to linear order in $\underline{I}_\perp$ by the eqs. (13) and (17). The theory for the former type of corrections is well known from the extensive literature on the subject (see e.g. refs.[1-6]) and shall not be considered here. In the present section we shall discuss the corrections of the latter type in the two cases of M1 and E2 moments. In a calculation of matrix elements between the eigenstates of $\underline{H}_{nn'}(\underline{I}_\perp)$ the renormalized operators, including these corrections, must be used.

The components $\mathcal{M}(1\mu)$ of a spherical tensor in the laboratory system are given[6] in terms of the components $\mathcal{M}(1\nu)'$ in the intrinsic system by

$$\mathcal{M}(1\mu) = \sum_\nu D^1_{\mu\nu}(\varphi,\vartheta,\psi)\,\mathcal{M}(1\nu)'. \tag{29}$$

We consider the renormalization, given in the notation of eq. (17) by,

$$\delta\mathcal{M}(1\mu, \underline{I}_\perp) \equiv \mathcal{M}(1\mu, \underline{I}_\perp) - \mathcal{M}(1\mu, 0). \tag{30}$$

Let us take first the magnetic moment

$$\underset{\sim}{\mu} = \underset{\sim}{J}_p + (g_p - 1)\,\underset{\sim}{S}_p + g_n\,\underset{\sim}{S}_n , \tag{31}$$

where $\underset{\sim}{J}$ is the total angular momentum of the protons, g_p (g_n) the proton (neutron) gyromagnetic ratio, and $\underset{\sim}{S}_p$ ($\underset{\sim}{S}_n$) the total proton (neutron) spin. From eqs. (17), (29), (30) and (31), we get, after some trivial transformations,

$$\delta\mu_\mu(\underset{\sim}{I}_\perp) = \sum_{\alpha\beta} g_\alpha{}^\beta \frac{D^1_{\mu\alpha} I_\beta + I_\beta D^1_{\mu\alpha}}{2} + \frac{1}{2} D^1_{\mu 0}\left(f_+{}^+ + (f_+{}^+)^\dagger - f_-{}^- - (f_-{}^-)^\dagger\right) \tag{32}$$

with

$$\langle n|f_\alpha{}^\beta|m\rangle \equiv \sum_{i\mathcal{P}} \frac{\langle n|\mu_\alpha|i\rangle\langle i|\underset{\sim}{J}|p\rangle}{E_i - E_m} \doteq \frac{\partial \omega_{pm}}{\partial I_\beta} , \tag{33a}$$

$$g_\alpha{}^\beta \equiv f_\alpha{}^\beta + (f_{-\alpha}{}^{-\beta})^\dagger , \tag{33b}$$

and

$$D^1_{\mu\pm}(\varphi,\vartheta,\psi) \equiv \mp\frac{1}{\sqrt{2}} D^1_{\mu(\pm1)}(\varphi,\vartheta,\psi). \tag{34}$$

The tensor $g_\alpha{}^\beta$ is the effective gyromagnetic tensor. The second term in eq. (32) gives, within the set $\mathcal{V}$, a contribution to the intrinsic magnetic moment of the γ -state. If we define

$$g_R \equiv \frac{1}{\mathcal{J}} \sum_i \frac{\langle 0|\mu_-|i\rangle\langle i|J_+|0\rangle}{E_i - E_0} , \tag{35a}$$

$$\langle n|\delta g^p_{\alpha\beta}|m\rangle \equiv \frac{1}{\mathcal{J}} \sum_i \frac{\langle n|\mu_\alpha - g_R J_\alpha|i\rangle\langle i|J_\beta|m\rangle}{E_i - E_p} , \tag{35b}$$

and make an expansion in $\delta\mathcal{J}^n_{\alpha\beta}$ and $\delta g^n_{\alpha\beta}$ of the tensor using eqs. (13), (14), (15), (23), (33), and (35), we get for the terms obtained in successive orders of this expansion

$$g^{(0)}{}_\alpha{}^\beta = g_R\,\delta_{\alpha\beta}, \tag{36a}$$

$$\langle n|g^{(1)}{}_\alpha{}^\beta|m\rangle = \langle n|\delta g^m{}_\alpha{}^\beta + (\delta g^n{}_{-\alpha}{}^{-\beta})^\dagger|m\rangle, \tag{36b}$$

$$\langle n|g^{(2)}{}_\alpha{}^\beta|m\rangle = -\frac{1}{\mathcal{J}} \sum_\gamma \langle n|\left(\delta g^m_{\alpha\gamma}\,\delta\mathcal{J}^{\gamma\beta} + \delta\mathcal{J}^{\gamma\beta}(\delta g^n_{(-\alpha)(-\gamma)})^\dagger\right)|m\rangle. \tag{36c}$$

In analogy with eq. (22c') we shall make the approximation

$$g^{(2)}{}_\alpha{}^\beta \approx -\frac{1}{2\mathcal{J}} \sum_\gamma \left(g^{(1)}_{\alpha\gamma}\,\delta\mathcal{J}^{\gamma\beta} + \delta\mathcal{J}^{\gamma\beta} g^{(1)}{}_{\alpha\gamma}\right) \tag{37}$$

corresponding to the assumptions $\delta \underline{g}^{n}_{\alpha\beta} \approx \delta \underline{g}^{n}_{\gamma\beta}$, and $\delta \underline{g}^{n}_{\alpha\beta} \approx (\delta \underline{g}^{n}_{(-\alpha)(-\beta)})^{\dagger}$, and we shall suppose, when using eq. (37), that $\underline{g}^{(1)}_{\alpha\beta}$ has a boson expansion similar to eq. (24).

We note that $\langle o | \underline{g}^{(1)}_{\alpha\beta} | o \rangle = 0$ by definition. The expression (35a) is simply the cranking formula for the collective gyromagnetic ratio[13]. An expression for the M1 transition matrix elements between the γ - and the ground state band, which is identical with that obtained from eq. (32) with the approximation $\underline{g}_{\alpha}{}^{\beta} \approx \underline{g}^{(0)}_{\alpha}{}^{\beta} + \underline{g}^{(1)}_{\alpha}{}^{\beta}$ was derived by Bes et al.[8] by assuming that the Coriolis interaction $\underline{V}_{Cor} = -\underline{I}_{\perp} \cdot \underline{J}_{\perp}/\mathcal{J}$ acts between the even- and odd-$\underline{K}$ bands, and treating this interaction in the approximation of first order perturbation theory. One will easily see that these two approximations are generally equivalent, provided terms corresponding to those included in $\underline{g}_{\alpha}{}^{\beta}$ are taken into account in the second term in eq. (32).

For the quadrupole moment $\underline{Q}_{\mu}$ we consider here only the collective part $\underline{Q}^{col}_{\mu}$, which is given by eq. (29) with $\underline{Q}^{col\,\prime}_{\mu} = \underline{Q}^{0} \delta_{\mu 0}$ where $\underline{Q}^{0}$ is constant. If we insert the expression for $\underline{Q}^{col}_{\mu}$ into eq. (8), we get a contribution to $\delta \underline{Q}_{\mu}(\underline{I}_{\perp})$ in second order of the expansion in $\underline{\omega}_{n\,m}$, when we write the operators $\underline{D}^{2}_{\mu 0}(\varphi, \vartheta, \psi)$ and $\underline{I}_{\perp}$ in the order suggested by the form of the right hand side in this equation. This contribution is associated with the (classically vanishing) operators

$$\underline{I}_{\alpha}\, \underline{D}^{2}_{\mu 0}\, \underline{I}_{\beta} - \frac{1}{2}\left(\underline{D}^{2}_{\mu 0}\, \underline{I}_{\alpha} \underline{I}_{\beta} + \underline{I}_{\alpha} \underline{I}_{\beta}\, \underline{D}^{2}_{\mu 0} \right) \qquad (38)$$

and can be written

$$\delta \underline{Q}_{\mu}(\underline{I}_{\perp}) = -\frac{Q^{0}}{8\mathcal{J}^{2}} \Big(6\, \underline{D}_{\mu 0} (Y_{+-} + Y_{-+}) + \sqrt{24}\, \left(\underline{D}^{2}_{\mu 2}\, Y_{++} + \underline{D}^{2}_{\mu(-2)}\, Y_{--} \right) - \sqrt{6}\, \left(\underline{D}^{2}_{\mu 1}\, I_{+} - \underline{D}^{2}_{\mu(-1)}\, \underline{I}_{-} \right)(Y_{+-} - Y_{-+}) \Big), \qquad (39)$$

where, in the limit of vanishing operators $\delta \underline{\mathcal{J}}^{n}_{\alpha\beta}$,

$$Y_{\alpha\beta} = \sum_{i} \frac{\langle n | \mathcal{J}_{\alpha} | i \rangle \langle i | \mathcal{J}_{\beta} | m \rangle}{(E_{i} - E_{n})(E_{i} - E_{m})} . \qquad (40)$$

The first two terms within the outer parentheses in eq. (39) are seen to involve simply a renormalization of the intrinsic matrix elements $\langle n | Q'_{\mu} | m \rangle$. From eqs. (23), (27), and (40) the correction to the matrix element $\langle n | Q'_{\mu} | m \rangle$ is expected to be roughly proportional to the corresponding matrix element of $\mathcal{J}_{\alpha\beta}$. Within the set $\mathcal{U}$ the third term in eq. (39) contributes only to the intraband matrix elements in the γ - band. For an arbitrary matrix element of the quadrupole tensor one can now prove the following†: If one adds to the matrix element between pure rotational state the band mixing contribution obtained in the Lipas approximation[1] with the reduced coupling matrix element given by the term $\underline{h}^{(1)\alpha\beta}$ in the expansion of $\underline{h}^{\alpha\beta}$ plus the contribution given by eqs. (39) and (40), then the result is identical with that obtained by a second order perturbation treatment of $\underline{V}_{Cor}$. Especially we get therefore in the approximation to our equations just described the same expression as obtained by Bès et al.[8]

Footnote

†A corresponding relation holds also if the non-collective part of the quadrupole tensor is included in the renormalization.

for the $E2$ transition probabilities between the γ- and the ground state band. In the formalism of these authors the renormalization of the matrix element $\langle 0|Q'_{-2}|\gamma\rangle$ by the contribution (39) is associated with the parameter y and it amounts according to their calculations to a reduction by about 1%. Thus it is practically negligible.

The present discussion shows, however, that a perturbation treatment of V_{Cor} is equivalent to our theory in the approximation, where second order terms like $h^{(2)\alpha\beta}$ and $g^{(2)}{}_{\alpha}{}^{\beta}$ are neglected†. We expect a priori this

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†With certain special additional assumptions a perturbation treatment of the particle plus rotor Hamiltonian, including the recoil term $V_{recoil} = J_1^2/2\mathcal{J}$ can be shown to give a contribution to $h^{\alpha\beta}$ with the product structure of eq. (22c')[14].

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to be a good approximation for the matrix elements between the ground state and the states $|\mu\rangle$, but not for the matrix elements between these states. Thus let us consider the Taylor expansion of the tensor $h^{\alpha\beta}$ in the Bohr model[15,25],

$$h_B^{\alpha\beta} \equiv \frac{1}{2}\left(\mathcal{J}_B^{-1}\right)^{\alpha\beta}(\beta)$$

$$= \frac{1}{2\mathcal{J}_{eq}} G^{\alpha\beta} - \frac{1}{2\mathcal{J}_{eq}^2}\sum_{\mu}\frac{\partial \mathcal{J}_B^{\alpha\beta}(\beta^{eq})}{\partial\beta_\mu}(\beta_\mu - \beta_\mu^{eq}) \qquad (41)$$

$$+\frac{1}{2\mathcal{J}_{eq}^3}\sum_{\mu\nu}\left(\sum_{\gamma}\frac{\partial \mathcal{J}_B^{\alpha\gamma}(\beta^{eq})}{\partial\beta_\mu}\frac{\partial \mathcal{J}_B^{\gamma\beta}(\beta^{eq})}{\partial\beta_\nu} - \frac{\mathcal{J}_{eq}}{2}\frac{\partial^2 \mathcal{J}_B^{\alpha\beta}(\beta^{eq})}{\partial\beta_\mu\partial\beta_\nu}\right)$$

$$\times(\beta_\mu - \beta_\mu^{eq})(\beta_\nu - \beta_\nu^{eq}).$$

Here $\mathcal{J}_{B\alpha\beta}$ denotes the inertial tensor (restriction of the Bohr) to the 1-2-plane, which can be expressed as a D_∞ -invariant tensor function of the deformations β_0 and $\beta_{\pm 2}$, β_μ^{eq} is the equilibrium value of β_μ ($\beta_{\pm 2}^{eq} = 0$) and

$$\mathcal{J}_{eq} \equiv \frac{1}{2}\mathcal{J}_{B+-}(\beta^{eq}) \qquad (42)$$

In the case of harmonic vibrations quantities of the form $\langle\mu|h^{\alpha\beta}|\nu\rangle - \delta_{\mu\nu}\langle 0|h^{\alpha\beta}|0\rangle$ rise from the third term in eq. (41)[51]. In this term the second derivative of $\mathcal{J}_B^{\alpha\beta}$ corresponds to a contribution of the type $h^{(1)\alpha\beta}$ while the product of first derivatives corresponds to a contribution of the type $h^{(2)\alpha\beta}$. With the hydrodynamical expression[15] for $\mathcal{J}_{B\alpha\beta}(\beta)$ these terms have a comparable size.

5. Hamiltonian

We assume now that the deformed model Hamiltonian H has the form

$$H = H_0 + H_{res} + \Delta E, \qquad (43)$$

where H_0 is the Hamiltonian of non-interacting Bogoliubov quasiparticles, H_{res} a residual interaction, and ΔE a constant term, included in order to give the ground state of H the energy zero. We write

$$H_0 = \sum_i E_i b_i^\dagger b_i, \qquad (44)$$

where b_i is the annihilation operator of a quasiparticle, and E_i the corresponding quasiparticle energy. H_o is supposed to be given by the standard Nilsson + BCS theory with a few modifications, which will be explained in sect. 7. The residual interaction is written in the form

$$H_{res} = -\frac{1}{24} \sum_{\iota\kappa\lambda\mu} v_{\iota\kappa\lambda\mu}\, b_\iota b_\kappa \left(b_\lambda b_\mu - 6 \langle 0 | b_\lambda b_\mu | 0 \rangle \right), \tag{45}$$

where the symbol b_ι - written with a Greek index - denotes either an annihilation or a creation operator. The summation in eq. (45) includes all combinations of such operators, and the interaction matrix element $v_{\iota\kappa\lambda\mu}$ is understood to be anti-symmetric in all its indices. The form of the expression (43) ensures that H_{res} does not contribute to the average single particle and pairing fields in the ground state $|0\rangle$.

In terms of H_o and H_{res} we can define the renormalized effective interaction H_{eff} for the Random Phase approximation (RPA). The formal definition is given in appendix C. We assume that an effective interaction, which accounts for the properties of the states $\mathcal{V}$, is energy independent and has the form of the pairing plus quadrupole force. In the notation explained at the end of appendix C it is written thus as

$$H_{eff} = -\frac{1}{2} \Big(\sum_{\mu=0,\pm2} \chi_\mu \hat{Q}_\mu^+ \hat{Q}_\mu + G_p \left(P_p^+ P_p + P_p P_p^+ \right) + G_n \left(P_n^+ P_n + P_n P_n^+ \right) \Big). \tag{46}$$

The symbol $\hat{Q}_\mu$ denotes a modified quadrupole moment, defined in analogy with the modified octupole moment introduced

in ref.[16],

$$\hat{Q}_\mu \equiv \sum_{ij} \langle i | \mathcal{Y}_{2\mu}(\hat{r}) | j \rangle\, a_i^+ a_j , \tag{47}$$

where $\mathcal{Y}_{2\mu}(\hat{r})$ is a spherical harmonic polynomium of the set of stretched spatial coordinates $\hat{r}$ defined by the identity

$$V_{osc}(\hat{r}) = \frac{1}{2} M \mathring{\omega}^2 \hat{r}^2 . \tag{48}$$

Here $V_{osc}(\hat{r})$ denotes the oscillator part of the Nilsson potential, M is the nucleon mass, and $\mathring{\omega} = (\omega_z \omega_\perp^2)^{\frac{1}{3}}$ where ω_z and $\omega_\perp$ are the oscillator frequencies. P_p and P_n are the monopole pair annihilation moments,

$$P_p \equiv \sum_{\substack{i>0 \\ \text{protons}}} a_{\bar{\imath}}\, a_i , \tag{49a}$$

$$P_n \equiv \sum_{\substack{i>0 \\ \text{neutrons}}} a_{\bar{\imath}}\, a_i . \tag{49b}$$

In eqs. (47) and (49) the operators a_i are annihilation operators of nucleons. The quantities χ_μ, G_p, and G_n are interaction constants. We demand that the spurious modes associated with particle number conservation shall be eliminated in the RPA. This implies[17] the BCS relations

$$G_p = \frac{2}{\sum_{\substack{i>0 \\ \text{protons}}} \frac{1}{E_i}} , \tag{50a}$$

$$G_n = \frac{2}{\sum_{\substack{i>0 \\ \text{neutrons}}} \frac{1}{E_i}} . \qquad (50b)$$

We define the normalized oscillating field associated with an excited state $|n\rangle$ by

$$F_n \equiv \frac{1}{4} \sum_{\iota\kappa\lambda\mu} v_{\iota\kappa\lambda\mu} \langle n | b_\iota b_\kappa | 0 \rangle b_\lambda b_\mu . \qquad (51)$$

The RPA equations can be expressed as equations for the energy E_n and field F_n. (See appendix C.) Writing eq. (44) in the short hand form

$$H_{eff} = -\frac{1}{2} \sum_\rho \chi_\rho Q_\rho^\dagger Q_\rho \qquad (52)$$

we have

$$F_n = \sum_\rho G(F_n^\dagger, Q_\rho^\dagger, E_n) \chi_\rho Q_\rho , \qquad (53)$$

$$G'(F_n^\dagger, F_n^\dagger, E_n) \equiv \left(\frac{dG(F_n^\dagger, F_n^\dagger, \omega)}{d\omega} \right)_{\omega = E_n} = 1 , \qquad (54)$$

where, for arbitrary field operators A and B,

$$G(A, B, \omega) \equiv \frac{1}{2} \sum_{ij} \left(\frac{A_{ij}^* B_{ij}}{E_i + E_j - \omega} + \frac{(A^\dagger)_{ij} (B^\dagger)_{ij}^*}{E_i + E_j + \omega} \right) . \qquad (55)$$

By a field operator we understand here an operator A with the structure

$$A = \sum_{\iota\kappa} \alpha_{\iota\kappa} b_\iota b_\kappa + \text{constant} , \qquad (56)$$

where $\alpha_{\iota\kappa} = -\alpha_{\kappa\iota}$ are constants. (In the present paper the constant term in eq. (56) - which may have an infinite value - is always insignificant. Linear relations between field operators, like eqs. (47), (49), (51), and (53), are understood as relations modulo a constant.) The component A_{ij} is defined by

$$A_{ij} \equiv \{ b_j, [b_i, A] \} . \qquad (57)$$

6. Matrix elements of $\delta\mathcal{T}_{\alpha\beta}$ and $g^{(1)}{}_{\alpha}{}^{\beta}$

From eqs. (23), (35b), and (36b), it is seen that a calculation of the matrix elements $\langle n | \delta\mathcal{T}_{\alpha\beta} | m \rangle$ and $\langle n | g^{(1)}{}_{\alpha}{}^{\beta} | m \rangle$ involves an evaluation of expressions with the structure

$$\langle n | A \frac{1}{H - \omega} B | m \rangle \qquad (58)$$

This is a special case within a class of matrix elements, which can be expressed as sums of Feynmann diagrams according to certain simple rules. The general theorems are proved in appendix B.

We use the theorems 2 and 3, and neglect all diagrams, which contain residual interaction lines. Hereby we obtain the graphical expressions shown in fig. 1. These expressions involve the oscillating fields of the excited states, defined in eq. (51). The precise rules for the interpretation of the diagrams are given in appendix A. In the evaluation of the expressions in fig. 1 we can take the advantage of the fact that in both expressions all possible time orders of the vertices appear. From a theorem, proved in appendix D, we get

$$\langle 0|\delta\mathcal{J}_{\alpha\beta}|n\rangle = G(J_{-\alpha},[J_\beta,\widehat{F_n^\dagger}],E_n) + G(J_{-\beta},[J_\alpha,\widehat{F_n^\dagger}],0), \tag{59a}$$

$$\langle m|\delta\mathcal{J}_{\alpha\beta}|n\rangle = G(F_m^\dagger,[J_\alpha,\widehat{[J_\beta,\widehat{F_n^\dagger}]}]+[J_\beta,\widehat{[J_\alpha,\widehat{F_n^\dagger}]}],E_m)$$
$$+ G(J_{-\alpha},[F_m,\widehat{[J_\beta,\widehat{F_n^\dagger}]}]+[J_\beta,\widehat{[F_m,\widehat{F_n^\dagger}]}],E_n-E_m)$$
$$+ G(J_{-\beta},[F_m,\widehat{[J_\alpha,\widehat{F_n^\dagger}]}]+[J_\alpha,\widehat{[F_m,\widehat{F_n^\dagger}]}],0). \tag{59b}$$

Here the operator $\widehat{A}$ is defined by the following equation for a field operator A, which appears in a diagram with the associated frequency ω.

$$[H_0,\widehat{A}] - \omega\widehat{A} = A. \tag{60}$$

For the commutator $[A_1,\widehat{A}_2]$ the sum of frequencies $\omega_1+\omega_2$ is used in this definition. Note that this commutator is a field operator itself. Our numerical calculation was based on the eqs. (59). (For further details, see sect. 7.)

Expressions for the matrix elements of $g^{(1)}_{\alpha\beta}$ are obtained by replacing J_α by $(\mu_\alpha - g_R J_\alpha)/\hbar$ in the eqs. (59). Like Bès et al.[8] we neglect the spin terms in eq. (31). Then we get

$$g^{(1)}{}_\alpha{}^\beta = \frac{1}{\hbar}\left(\delta\mathcal{J}^{(p)}{}_\alpha{}^\beta - \delta\mathcal{J}_\alpha{}^\beta\right), \tag{61}$$

where the matrix elements of $\delta\mathcal{J}^{(p)}_{\alpha\beta}$ are obtained by a summation of the proton diagrams in fig. 1. (Bès et al. included in their calculation only the diagrams in fig. 1a with $F_\gamma^\dagger$ as the downmost or upmost vertex.)

When the spin terms are neglected, and we neglect also the contribution from the second term in eq. (32), the intrinsic gyromagnetic ratio g_k of the γ-state is given by

$$g_k = \frac{\langle\gamma|\mu_3|\gamma\rangle}{2} = \frac{\langle\gamma|J_{p3}|\gamma\rangle}{2} = G^{(p)'}(F_\gamma^\dagger,F_\gamma^\dagger,E_\gamma), \tag{62}$$

where $G^{(p)'}(F_\gamma^\dagger,F_\gamma^\dagger,E_\gamma)$ is the proton contribution to the derivative in eq. (54).

We shall discuss now the relation between the equations

(59) and those obtained from an adiabatic treatment[18] of the effective Hamiltonian defined in sect. 5. Thus we consider a variation of the quasiparticle Hamiltonian

$$H_o \to H^{(\omega)}(\beta, \Delta, \underline{\lambda})$$

$$= H_o - \sum_{\underline{\mu}} \hat{Q}^{\dagger}_{\underline{\mu}} dq_{\underline{\mu}} - (P_p^{\dagger} + P_p) d\underline{\Delta}_p - (P_n^{\dagger} + P_n) d\underline{\Delta}_n \qquad (63)$$

$$- N_p \, d\underline{\lambda}_p - N_n \, d\underline{\lambda}_n$$

$$\equiv H_o - \sum_{\underline{p}} \tilde{Q}_p^{\dagger} dq_{\underline{p}}$$

under the constraint of conservation of the average of the particle numbers $\underline{N}_p$ and $\underline{N}_n$ in the quasiparticle vacuum. The normal modes of adiabatic harmonic vibrations in the coordinates $\underline{q}_{\underline{p}}$ are derived from the restoring force matrix

$$\mathcal{V}_{pq} = G(\tilde{Q}_p^{\dagger}, \tilde{Q}_q^{\dagger}, 0) - \sum_{\underline{r}} G(\tilde{Q}_p^{\dagger}, \tilde{Q}_r^{\dagger}, 0) \chi_r \, G(\tilde{Q}_r^{\dagger}, \tilde{Q}_q^{\dagger}, 0) \qquad (63)$$

and the mass tensor, given by the cranking formula

$$\mathcal{M}_{pq} = \frac{1}{2} G''(\tilde{Q}_p^{\dagger}, \tilde{Q}_q^{\dagger}, 0). \qquad (64)$$

The condition of particle number conservation gives the linear constraints

$$\sum_{\underline{p}} G(N_p, \tilde{Q}_p^{\dagger}, 0) dq_{\underline{p}} = \sum_{\underline{p}} G(N_n, \tilde{Q}_p^{\dagger}, 0) dq_{\underline{p}} = 0 \qquad (65)$$

The coupling constants $\underline{\chi}_p$ associated with the variables

$\underline{q}_{\underline{r}}$, $\underline{\Delta}_p$, and $\underline{\Delta}_n$, are equal to $\underline{\chi}_{\underline{r}}$, $\underline{G}_p$, and $\underline{G}_n$, respectively. These are the only ones, which contribute to the sum in eq. (63), due to the constraints (65). For given values of the coordinates $\underline{q}_{\underline{p}}$ the adiabatic inertial tensor is given by

$$\mathcal{J}^{(\omega)}_{\alpha\beta}(q) = G(J_{\alpha}, J_{\beta}, 0)_{H_o = H^{(\omega)}(q)}. \qquad (66)$$

The difference tensor $\delta\mathcal{J}^{(\omega)}_{\alpha\beta}(q)$ can be defined in this case as $\mathcal{J}^{(\omega)}_{\alpha\beta}(q) - \langle 0_\nu | \mathcal{J}^{(\omega)}_{\alpha\beta}(q) | 0_\nu \rangle$ where $|0_\nu\rangle$ is the vacuum of vibrational quanta.

Now the cranking formula (66) corresponds to the diagrams in fig. 2. We get the derivative $(\partial \mathcal{J}^{(\omega)}_{\alpha\beta} / \partial q_{\underline{p}})_{dq=0}$ by inserting a vertex with the field $-\partial \underline{H}^{(\omega)}(q) / \partial q_{\underline{p}} = \tilde{Q}^{\dagger}_{\underline{p}}$ and zero frequency into an arbitrary particle line with an arbitrary time position relative to the original vertices and making a summation over all diagrams thus obtained. (Appendix D.) Hence the derivative is given simply by the diagrams in fig. 1a with $\underline{E}_n = 0$, $\underline{F}_n = \tilde{Q}_p$. Repeating the argument we see that a similar relation exists between the second derivative $(\partial^2 \mathcal{J}^{(\omega)}_{\alpha\beta} / \partial q_{\underline{p}} \partial q_{\underline{q}})_{dq=0}$ and the expression in fig. 1b. Thus from a Taylor expansion of $\mathcal{J}^{(\omega)}_{\alpha\beta}(q)$ it is seen, that if $\underline{F}_n$ is replaced by the adiabatic oscillating field

$$\tilde{F}_n = \sum_{\underline{p}} (dq^n_{\underline{p}})^* Q_p \qquad (67)$$

where $dq^n_{\underline{p}}$ is the zero-point amplitude of $q_{\underline{p}}$ in the normal mode $\underline{n}$, then the eqs. (59) give in the limit $\underline{E}_n \to 0$

the matrix elements of $\delta \mathcal{H}^{(0)}_{\nu\mu}(q)$ in the approximation, where terms in the Taylor expansion of higher than quadratic order are neglected.

The fields F_γ and $\tilde{F}_\gamma$ are both proportional to $\hat{Q}_{-2}$, and the normalizations coincide asymptotically in the limit $E_\gamma \to 0$. For the β-vibrations the relation is more involved: The field F_β contains two terms of the asymptotic form $\Lambda E_\beta^{-\frac{3}{2}}(P - P^\dagger)$ in the limit $E_\beta \to 0$. Here P represents P_p or P_n, and Λ is a constant. The field $\tilde{F}_\beta$ is asymptotically identical with the field F'_β obtained by replacing this term by $(\Lambda/2\Delta)E_\beta^{-\frac{1}{2}} N$. The fields F_β and F'_β give identical matrix elements $\langle o|\delta \mathcal{H}_{+-}|\beta\rangle$ and $\langle\beta|\delta \mathcal{H}_{-+}|\gamma\rangle$ when inserted into the eqs. (59), but the matrix elements $\langle\beta|\delta \mathcal{H}_{+-}|\beta\rangle$ differ by a term $(\Lambda^2/\Delta)E_\beta^{-3}(\partial \mathcal{H}^{(0)}_{+-}/\partial\Delta)_{dq=0}$ when calculated respectively from F_β and F'_β. The singularity of this term for $E_\beta \to 0$ is seen to be stronger than the normal singularity of the order E_β^{-1} associated with the increase of the zero-points amplitudes with decreasing energy.

Thus for all matrix elements except $\langle\beta|\delta \mathcal{H}_{+-}|\beta\rangle$ the eqs. (53), (54), and (59) give the adiabatic result in the limit of vanishing phonon energies. The divergence of the matrix element $\langle\beta|\delta \mathcal{H}_{+-}|\beta\rangle$ in this limit has to do with the way particle number conservation is treated in the Bès theory for the β-vibrations[17], which is applied here in its essential traits. This theory ensures $\langle o|N|\beta\rangle = 0$ but not $\langle\beta|N|\beta\rangle = \langle o|N|o\rangle$. In fact the difference $\langle\beta|N|\beta\rangle - \langle o|N|o\rangle$ contains a divergent term similar to the term in $\langle\beta|\delta \mathcal{H}_{+-}|\beta\rangle$ just considered, namely $(\Lambda^2/\Delta)E_\beta^{-3}(\partial N^{(0)}/\partial\Delta)_{dq=0}$, where $N^{(0)}$ denotes the expectation value of N in the ground state of $H^{(0)}(q)$. In the theory of Soloviev[19] the chemical potential

is adjusted for the single phonon state, so that $\langle\beta|N|\beta\rangle = \langle o|N|o\rangle$ is satisfied. Hereby, however, the relation $\langle o|N|\beta\rangle = 0$ is destroyed†. Soloviev found that for the actual energies E_β

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†One could note in this connection that a theory, where both $\langle o|N|\beta\rangle = 0$ and $\langle\beta|N|\beta\rangle = \langle o|N|o\rangle$ might be constructed by treating the chemical potential as a matrix analogous to the matrices ω^α_{nm} of sect. 2.

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the difference $\langle\beta|N|\beta\rangle - \langle o|N|o\rangle$ is usually negligible. Accordingly we assume in the present work that the Bès theory is sufficiently accurate.

The formula for the matrix elements $\langle o|\delta \mathcal{H}_{\nu\mu}|\mu\rangle$ obtained by Pavlichenkov corresponds to eq. (59a) with $E_\mu = 0$, i.e. to the adiabatic formula with the non-adiabatic normalization of F_μ. His expression for the matrix elements $\langle\mu|\delta \mathcal{H}_{\nu\beta}|\mu\rangle$ is in principle equivalent to eq. (59b). In both cases Pavlichenkov makes an analytic approximation, where he utilizes the simple properties of the anisotropic oscillator single particle potential.

7. Details of the calculation

In the Nilsson calculation we used the standard parameters $\hbar\omega = 41/A^{\frac{1}{3}}$ MeV, $\kappa_p = \kappa_n = 0.0637$, $\mu_p = 0.6$, and $\mu_n = 0.42$. No ℓ_4-term was included, and the Nilsson Hamiltonian was diagonalized within each major shell using the

optimally stretched oscillator basis[20]. The quadrupole deformations δ were taken from ref.[21]. Values of the gap parameters Δ_p and Δ_n were calculated from the empirical masses[22] by means of a quadratic fit to five nuclei with successive numbers of protons (neutrons), equally distributed around the even nucleus considered, and the chemical potentials were determined so as to satisfy $\langle N_p \rangle = Z$ and $\langle N_n \rangle = N$ in the quasiparticle vacuum.

We included in the calculation the proton levels in the major shells with N = 3,4, and 5, and those originating in the spherical subshells with $(N,j) = (6, \frac{13}{2} - \frac{9}{2})$, $(7, \frac{15}{2})$, and $(8, \frac{17}{2})$. For the neutrons the analogous levels in the major shells with N=4-9 were taken into account. All these levels are situated within an interval of the size $\approx 3\hbar\omega$ around the Fermi level. Only the $\Delta N = 0$ matrix elements of F_μ and J_α were included. Including in a test case all the matrix elements or single particle levels within an interval of the order $5\hbar\omega$ we found that the matrix elements of $\delta\mathcal{J}_{\alpha\beta}$ were changed by $\lesssim$ 10%. In the vicinity of the Fermi level we adjusted the Nilsson levels in accordance with the reduced empirical energies compiled in ref.[23].

Given the quasiparticle energies E_i, the pairing interaction strengths G_p and G_n are determined by eq. (50). The constants χ_μ of the quadrupole interaction were chosen for each nucleus such that the experimental phonon energies[28] were reproduced. In a few cases we had to estimate the energy E_β. These are: ^{166}Er : E_β = 1.25 MeV, $^{186-188}Os$: 1.1 MeV. Both values were chosen on the basis of the systematics in neighbouring nuclei. In the adiabatic calculation χ_0 and $\chi_{\pm 2}$ were determined from the adiabatic equations.

The cranking model moment of inertia $\mathcal{J}$ was taken as an empirical parameter, derived by means of eq. (25) from the effective moment of inertia $\mathcal{J}_{eff}$ of the intrinsic ground state. The latter quantity was obtained by fitting the two lowest excited levels in the ground state band by the expression $E_I = (1/2\mathcal{J}_{eff})I(I+1) + B(I(I+1))^2$. The collective gyromagnetic ratio g_R was derived in the analogous way from the empirical magnetic moments of the lowest 2^+ state compiled in ref.[24].

Thus our calculation contains no adjustable parameters.

In ref.[16] we gave for the case of the octupole force an argument for the assumption, that when the fields of the interaction are expressed in terms of the stretched coordinates $\hat{r}$ as in eq. (47), then the coupling constant associated with different projections of the phonon angular momentum should be approximately equal with a small deformation dependent correction term. The formula corresponding to eq. (2.20) in the quoted ref. has the following form for the quadrupole interaction,

$$\chi_\mu = \frac{p}{A^2}\left(1 + \sqrt{\frac{5}{4\pi}}\,\frac{2-\mu^2}{2}\,\beta\right), \qquad (68)$$

where p (practically) is a constant. From fig. 3 eq. (68) is seen to reproduce relatively well the qualitative situation, like in the octupole case, although it is not sufficiently accurate for the purpose of quantitative predictions. The largest derivations from eq. (68) are seen to appear for the constant χ_0. Probably this has to do with the admixtures of pairing vibrations in the $K = 0$ quadrupole vibrational mode.

For the numerical evaluation of the expressions (59) we constructed a general algorithm, which calculates the components of the field $[\underline{A}_1, \underline{\hat{A}}_2]$ from those of $\underline{A}_1$ and $\underline{A}_2$ for a given frequency of the field $\underline{A}_2$. This method is especially powerful for the sum of four-vertex diagrams in fig. 1b. Thus it is seen, that by means of eq. (59b) the sum, according to fig. 1b, over four independent sets of the single particle quantum numbers and 72 different diagrams† is reduced to a few successive

Footnote

†Some of these can be combined to a single term.

applications of the algorithm just mentioned, and a trivial algorithm for the G-function (55). The algorithm for $[\underline{A}_1, \underline{\hat{A}}_2]$ involves only three nested loops through the single particle quantum numbers.

It appeared in the calculations that the second and third term in eq. (59b) always have the same sign, which is the opposite of that of the first term. The total value of the expression is a relatively small difference between these terms.

8. Results and discussion

We have calculated the matrix elements of $\underline{\delta \mathcal{J}}_{\alpha\beta}$ between the states ν for 15 even isotopes of the elements $Sm - Os$. The results are shown in table 1.

The quantity shown in column 3 of the table is the reduced A-parameter $1/2\mathcal{J}$ obtained from the empirical value in column

2 by means of eq. (25). The reduction is seen to amount to about 10%. Obviously, it is the quantity $\mathcal{J}$ rather than $\mathcal{J}_{eff}$ which is relevant for a comparison with empirical data of theoretical cranking model calculations of the ground state moment of inertia. The 10% difference between $\mathcal{J}$ and $\mathcal{J}_{eff}$ is significant for an empirical test of corrections to the simple cranking model with noninteracting quasiparticles. (Cf. e. g. ref. [26].)

For a few selected nuclei we have made an adiabatic calculation as described in sect. 6†. It is seen that the

Footnote

†Note that the coordinates Δ_p and Δ_n are treated here as dynamical variables on an equal footing with the coordinate β. This corresponds to the way the pairing field is treated in the non-adiabatic case. Bès has given in the appendix of ref. [27] a different formalism, where Δ_p and Δ_n are related to β by the condition of static self-consistency of the pairing field. Both schemes are equivalent in the limit $\underline{E}_\beta \to 0$, but the present one yields a closer approximation to the RPA for finite energies.

adiabatic results are close to the non-adiabatic ones for the matrix elements $\langle 0|\delta \mathcal{J}_{\alpha\beta}|\mu\rangle$, while for the matrix elements $\langle \mu|\delta \mathcal{J}_{\alpha\beta}|\nu\rangle$ there are large deviations between the two approximations, both with respect to the magnitude and the sign.

From the non-adiabatic matrix elements of $\underline{\delta \mathcal{J}}_{\alpha\beta}$ we have calculated the matrix elements of the tensor $\underline{h}^{\nu\beta}$ using the

approximation for $\underline{h}^{(2)\alpha\beta}$ described in sect. 3. As the size of matrix element $\langle\mu|\delta\mathcal{J}_{\alpha\beta}|\nu\rangle$ can be comparable to that of the matrix elements $\langle 0|\delta\mathcal{J}_{\alpha\beta}|\nu\rangle$, we have included in the approximation for $\underline{h}^{(2)\alpha\beta}$ the products of the quadratic term in eq. (24). The results are shown in figs. 4-8. (Solid lines.) In all the figs. we show also the matrix elements of the term $\underline{h}^{(1)\alpha\beta}$ (dashed lines). As mentioned above this corresponds to the approximation obtained by treating the Coriolis force by second order perturbation theory. The $\underline{h}^{(2)\alpha\beta}$-term is seen to give appreciable corrections in all cases, except for the major part of the matrix elements $\langle 0|\underline{h}_{-2}|\gamma\rangle$.

The off-diagonal matrix elements $\langle 0|\underline{h}_0|\beta\rangle$, $\langle 0|\underline{h}_{-2}|\gamma\rangle$, and $\langle\beta|\underline{h}_{-2}|\gamma\rangle$ are compared to experimental data in figs. 4-6. A simultaneous fit of these three matrix elements to the empirical $E2$ branching ratios was made for three nuclei: ^{158}Gd [3] and $^{182\text{-}184}W$ [4]. Due to the comment given later[30] to the ^{184}W-data we have not, however, included these data in figs. 4-6. The values for ^{158}Gd and ^{182}W are indicated by crosses. Rud et al.[2] made an analysis of $B(E2)$-ratios between the β- and ground state bands in ^{152}Sm and $^{154\text{-}156}Gd$ in terms of two parameters associated with the matrix elements $\langle 0|\underline{h}_0|\beta\rangle$ and $\langle\beta|\underline{h}_{-2}|\gamma\rangle$. The values obtained by this analysis are shown as triangles. The circles in figs. 4 and 5 represent matrix elements derived from reported values of the Lipas parameter $\underline{z}_0$ [31-34] or $\underline{z}_2$ [30,35-39] †. Here only data, from

† Ref.[35] contains an extensive compilation of previous data for the nuclei ^{152}Sm - ^{166}Er as well as data published later in a number of individual papers from the Nashville group.

which the matrix element of $\underline{h}_{\alpha\beta}$ could be derived by means of an experimentally known $B(E2, 0_o \rightarrow 2_{\beta \text{ or } \gamma})$ have been taken into account. For $\underline{z}_0$ we have included only such nuclei, where the same value gives a consistent fit to several branching ratios. For $\underline{z}_2$ we have used in all cases the value, which fits the ratio $B(E2, 3_\gamma \rightarrow 2_o) / B(E2, 3_\gamma \rightarrow 4_o)$.

The agreement of the calculated values of $\langle 0|\underline{h}_0|\beta\rangle$ and $\langle 0|\underline{h}_{-2}|\gamma\rangle$ with the experimental data (figs. 4 and 5) is satisfactory. The $\underline{h}^{(1)\alpha\beta}$-contribution is generally the main contribution, but we note that the $\underline{h}^{(2)\alpha\beta}$-term tends to improve the fit.

The empirical knowledge of the matrix element $\langle\beta|\underline{h}_{-2}|\gamma\rangle$ is rather scarce and uncertain. In addition to the evidence obtained from the works mentioned above we can obtain an upper limit for this matrix element in the nucleus ^{170}Yb from the fact that here the 2^+ members of the β- and the γ- band are separated only by 7 keV. This inhibits a value of $\langle\beta|\underline{h}_{-2}|\gamma\rangle$ larger than 0.5 keV (Square point in fig. 6).

Our theory gives for most of the nuclei a value of $\langle\beta|\underline{h}_{-2}|\gamma\rangle$ between 0 and 1 keV. Thus the large values obtained by Rud et al. for ^{152}Sm and ^{154}Gd are definitely outside the scope of the model. With respect to these numbers it should be noted that they were obtained by an analysis which included only transitions between the β- and the ground state band. A similar analysis[35] of the γ- band does not give the same result†. It seems therefore that a coupling to higher

† Furthermore the formalism of ref.[2] may be criticized, cf. ref.[5].

excited states in the spectrum must be responsible for part of the effect considered by Rud et al. The experimental matrix element in ^{182}W is about four times larger than the calculated value. However, the order of magnitude is here in agreement with the theretical value.

The calculated differences $\langle\mu|h_0|\mu\rangle - \langle 0|h_0|0\rangle$ i.e. the change δA_μ of the A-parameter from the ground state band to an excited band are displayed in figs. 7 and 8.

The experimental values in fig. 7 were derived from an analysis of the excitation energies in the β- band similar to that made for the ground state band (sect. 7). Only nuclei where at least three members of the β- band are known, are included. The systematic behaviour of the empirical δA_β through the mass region is seen to be reproduced to some extend. The variation of the calculated values, which is similar to that of the empirical data, originates essentially in the variation of the matrix element $\langle\beta|h_0^{(1)}|\beta\rangle$ (Cf. the dashed curve.) When the $h_0^{(2)}$ term is included, the theoretical curve is pushed toward the positive side. This seems not to agree with the data, which have about an equal variation from zero in the positive and negative direction. For comparison we show in fig. 7 the results obtained by Pavlichenkov[7] on the basis of the harmonic oscillator single particle model.

The calculated values of δA_γ exhibit a much smoother behaviour than the δA_β due to the smaller variation of the matrix elements $\langle\gamma|h_0^{(1)}|\gamma\rangle$. We have omitted in fig. 8 the comparison with empirical data. One will understand the reason by considering the plot in fig. 9. Here four γ- bands known with at least four levels are shown in a plot of $(E_I - E_{I-1})/2I$

vs. $I^2 - 4$. It is obvious that no common parametrization in terms of I simultaneously fits the pattern of all these four bands. Therefore the extraction of the small difference δA_γ from the data is very ambiguous. From a consideration of differences between the two first members of the observed γ- bands it seems, however, that the variation of δA_γ is larger than predicted by our model. (These differences can be quite unreliable, however, as seen from fig. 9.) The well determined value in ^{166}Er is off the calculation by -3 keV.

The calculated matrix elements of the tensor $g^{(1)}{}_\alpha{}^\beta$ as given by eq. (61) are shown in table 3 together with the collective gyromagnetic ratio g_R used in the calculation, and the calculated difference $g_K - g_R$ for the γ- state.

The $g^{(2)}{}_+{}^+$ - correction to g_R is of the order 0.01 and thus comparable to or smaller than the experimental uncertainty (0.01 - 0.03).

The absolute difference $|g_K - g_R|$ has been measured in ^{166}Er [41,42]. The calculated value agrees well with the experimental $|g_K - g_R| \approx 0.10$.

The gyromagnetic tensor $g_\alpha{}^\beta$ is given by the eqs. (36) in terms of the matrix elements in tables 1 and 3. In order to test the calculated matrix elements against experiment we have evaluated the reduced M1 transition probabilities between the 2^+ - states, using

$$\mathcal{M}(M1\mu) = \sqrt{\frac{3}{4\pi}}\left(\sum_{\alpha\beta} g_\alpha{}^\beta \frac{D^1_{\mu\alpha} I_\beta + I_\alpha D^1_{\mu\alpha}}{2} + D^1_{\mu 0}\, g_K I_3\right). \quad (69)$$

The mixing of the three bands considered was taken into account in the approximation of first order perturbation theory. We used the calculated matrix elements of $\underline{h}^{\alpha\beta}$ and the empirical energy differences between the 2^+ states. The results are shown in table 4.

With a few exceptions our calculated values have the observed order of magnitude, $\approx 10^{-4}(e/2Mc)^2$. In the Hf and W isotopes our results for the quantity $B(M1, 2_\gamma \to 2_0)$ agree considerably better with the experimental data than those obtained by Bès et al. [8]. For this transition our calculated matrix elements $\langle 2_0 \| \mathcal{M}(M1) \| 2_\gamma \rangle$ have also a dominating negative sign like the observed ones[43].

A very detailed fit to the data should not be expected from the present model, as all the basic matrix elements $\langle n | g_\alpha^{(\pi)} \underline{\beta} | m \rangle$ are highly sensitive to the detailed choice of parameters. Furthermore we have neglected the spin part of the magnetic moment operator as well as possible contributions due to polarization effects. Both of these might play a significant role. Polarization effects could also modify our results for the matrix elements of $\delta \mathcal{I}_{\alpha\beta}$. Here, however, they are probably minor important.

Using the same scheme as for the $B(M1)$ we have calculated also the values of $B(E2, 0_0 \to 2_\mu)$. This may be taken essentially as a test of the RPA wave functions. The matrix elements $\langle 0 | Q'_{-\mu} | \mu \rangle$ were obtained by the standard method of the RPA, the matrix element $\langle \beta | Q'_{-2} | \gamma \rangle$ taken equal to zero, and all intrinsic states assumed to have a common quadrupole moment Q°. The latter parameter was taken from ref. [21].

In accordance with previous investigations[49] we found it necessary to use different polarization charges for the β- and the γ- vibrational mode. The results obtained for $e^\beta_{pol} = 0.0$ and $e^\gamma_{pol} = 0.3$ are shown in table 5. It is seen that this choice leads to a good over-all agrement with the data†.

Footnote

†In ref. [49] $e^\beta_{pol} = 0.3$ was used and found to lead to an over-estimate of the $B(E2, 0_0 \to 2_\beta)$ values. The discrepancy was reduced, when a spin-quadrupole term [50] was included in the effective interaction.

The large values of the matrix element $\langle \rho | \delta \mathcal{I}_{+-} | \rho \rangle$ found in the adiabatic calculation (table 2) might seem surprising in the light of the simple interpretation of this matrix element as proportional to a second derivative of the moment of inertia. One should note, however, that in the same calculation the matrix element $\langle 0 | \delta \mathcal{I}_{+-} | \rho \rangle$ is also relatively large, only about 3 times smaller than $2\mathcal{I}$. Thus the ratios $\langle 0 | \delta \mathcal{I}_{+-} | \rho \rangle / 2\mathcal{I}$ and $\langle \rho | \delta \mathcal{I}_{+-} | \rho \rangle / \langle 0 | \delta \mathcal{I}_{+-} | \rho \rangle$ are actually comparable, in accordance with a smooth variation of the moment of inertia. (Note that the direction in the $(\beta, \Delta_p, \Delta_n)$-space, in which the derivatives are taken, differs from the direction of an adiabatic variation of the coordinate β due to the dynamical treatment of the coordinates Δ_p and Δ_n.)

9. Conclusion

We have formulated a non-adiabatic, microscopic theory for the rotation-vibration interaction in well-deformed, even

nuclei. On the basis of this theory we have calculated various measurable quantities. A good agreement with the data was found for the quantities $\langle o|\underline{h}_o|\beta\rangle$, and $\langle o|\underline{h}_{-2}|\gamma\rangle$, and a qualitative agreement for the quantities $\delta\underline{A}_\beta$, $B(M1, 2_\beta \to 2_o)$ and $B(M1, 2_\gamma \to 2_o)$. For the quantities $B(M1, 2_\beta \to 2_\gamma)$, $\langle\beta|\underline{h}_{-2}|\gamma\rangle$, and $\delta\underline{A}_\gamma$, the data are either non-existent or their interpretation so ambigous that no decisive conclusion can be drawn. The latter quantity seems, however, to show a larger variation through the mass region than predicted by our model.

Our comparison with the results obtained within the framework of an adiabatic description of the vibrational motion showed that for the matrix elements $\langle\mu|\delta\mathcal{T}_{\alpha\beta}|\nu\rangle$ the adiabatic value cannot be taken even as a relatively good approximation to the non-adiabatic one. This seems to indicate that some precaution must be taken with respect to the believability of results obtained by adiabatic methods as far as branching ratios of electromagnetic transitions are concernced.

The large deviations between the adiabatic and non-adiabatic results for the matrix elements $\langle\mu|\delta\mathcal{T}_{\alpha\beta}|\nu\rangle$ indicate also a larger sensitivity to the detailed assumptions of the model for these matrix elements than for the matrix elements $\langle o|\delta\mathcal{T}_{\alpha\beta}|\mu\rangle$. We have presented here calculations based on the traditional choice of the pairing plus quadrupole effective interaction. The inclusion of other terms like e.g. a quadrupole-pairing term, a spin-quadrupole term, etc., could probably modify our results for the matrix elements $\langle\mu|\delta\mathcal{T}_{\alpha\beta}|\nu\rangle$.

In sect. 3 we made a rather crude approximation for the term $\underline{h}^{(2)\alpha\beta}$. The fact that for the quantity $\delta\underline{A}_\beta$ the

inclusion of this term tends to make the agreement with experiments worse rather than the opposite could indicate the necessity of a more accurate treatment of this term. An accurate treatment is possible in principle on the basis of eq. (22c), but the practical problems are tremendous. Thus the lowest order diagrams contributing to the matrix elements $\langle\mu|\underline{h}^{(2)\alpha\beta}|\nu\rangle$ contain six particle lines, and the time order of the vertices is only partially free.

The present work was initiated by a proposal from Prof. A. Bohr. His continous interest and valuable suggestions during the work are highly appreciated. Discussions with Dr. G.B. Hagemann on experimental data were extremely useful in connection with sect. 8. We appreciate also profitable discussions with Prof. B.R. Mottelson and Dr. I. Hamamoto as well as the financial support through a Nordita-fellowship in the period 1971-74.

Appendix A

(Diagram rules)

The rules of the presently applied version of the Feynmann diagram language are given below.

An arbitrary example of a diagram is shown in fig. 10.

The particle lines (solid) represent Bogoliubov quasiparticles. In order to evaluate the diagram one assigns a direction to each closed loop of particle lines, and attributes an operator $\underline{b}_{line} = \underline{b}_i$ to each upgoing line, and $\underline{b}_{line} = \underline{b}_i^\dagger$ to each downgoing line. Each downgoing line and closed loop contributes to the value of the diagram with the factor -1.

An external vertex (short, dashed line) is characterized by a field operator (cf. sect. 5) $\underline{F}$ and a frequency ω. The sum of external frequencies is always zero. Each external vertex contributes in an obvious symbolic notation with the factor

$$vertex(F) = \{ b_{out}, [F, b_{in}^\dagger] \}. \tag{70}$$

If the vertex is inserted in a loop with only a single particle line, a factor $\frac{i}{2}$ must be added. The constant term in the expression (56) is associated with a disconnected vertex.

The interaction lines (long, dashed lines) represent the residual interaction (45). Each interaction line gives a factor

$$interaction\ line = \frac{1}{4} \sum_{i\kappa\lambda\mu} V_{i\kappa\lambda\mu} \times vertex(\underline{b}_i \underline{b}_\kappa)_{left} \times vertex(\underline{b}_\lambda \underline{b}_\mu)_{right}. \tag{71}$$

A diagram obtained by exchanging the end points of two particle lines between the vertices of an interaction line is equivalent with the original one.

For each vertical level between the interaction lines and external vertices the expression for the diagram contains an energy denominator, given by

$$energy\ denominator = \sum_{\substack{crossing \\ lines}} \varepsilon_i - \sum_{\substack{vertices \\ below}} \omega. \tag{72}$$

If two particle lines go between the same two vertices, either external or belonging to an interaction line, a factor $\frac{1}{2}$ must be added.

The value of the diagram is obtained by a summation over all combination of the single particle quantum numbers i assigned to each particle line.

A diagram must not contain an unlinked part without external vertices, and not a part without external vertices, which is linked to the rest of the diagram only by a single interaction line.

In appendix B and C we draw a box to indicate the sum over a certain class of diagrams, specified in the text.

Appendix B

(Graphical expressions for matrix elements)

We consider operators of the form

$$G = A_0 \frac{1}{H-\omega_1} A_1 \cdots A_{N-1} \frac{1}{H-\omega_N} A_N \,, \tag{73}$$

where $A_0, \ldots, A_N$ are field operators.

Theorem 1. The ground state expectation value $\langle 0|G|0\rangle$ is given by the expression in fig. 11a, where the sum, represented by the box, includes all non-equivalent diagrams, which satisfy the rules of appendix A.

Proof (sketched). In the expression for the expectation value we insert the Brioullin-Wigner expansion[12] of the ground state wave function, and make the Taylor expansion

$$\frac{1}{H-\omega} = \frac{1}{H_0-\omega} - \frac{1}{H_0-\omega}(H_{res}+\Delta E)\frac{1}{H_0-\omega} + \cdots . \tag{74}$$

Taking all contractions we get linked as well as unlinked diagrams. Here those diagrams, which have an unlinked part without external vertices that is entirely above or below the upmost external vertex, cancel the terms of higher than zeroth order in ΔE in the same way as for the ground state energy[12]. Those unlinked parts without external vertices, which cross the level of the upmost external vertex, give a factor, which is cancelled by the normalization of the Brioullin-Wigner wave function. Parts without external vertices, which are linked to the rest of the diagram by a single interaction line, are cancelled by the subtracted average term in eq. (45). q.e.d.

Theorem 2. The matrix element $\langle 0|G|n\rangle$ between the ground state and an excited state $|n\rangle$ with energy E_n is given by the expression in fig. 11b, where F_n is the oscillating field of the state. (Eq. (51).) The vertex $F_n^\dagger$ can have all vertical ('time') positions relative to the vertices $A_0, \ldots, A_N$. The sum, indicated by the box, is restricted by the following rule for the diagrams included: $F_n^\dagger$ must be linked to at least one of the vertices $A_0, \ldots, A_N$, and it must not be possible to cut it from the rest of the diagram by the removal of a single interaction line.

Proof. We introduce an auxiliary field operator B and write

$$\begin{aligned} S(\omega) \equiv \langle 0| \Big(& A_0 \frac{1}{H-\omega_1} A_1 \cdots A_{N-1} \frac{1}{H-\omega_N} A_N \frac{1}{H-\omega} B \\ & + A_0 \frac{1}{H-\omega_1} A_1 \cdots A_{N-1} \frac{1}{H-\omega_N} B \frac{1}{H-\omega_N+\omega} A_N \\ & + \cdots + B \frac{1}{H+\omega} A_0 \frac{1}{H-\omega_1+\omega} A_1 \cdots A_{N-1} \frac{1}{H-\omega_N+\omega} \Big) |0\rangle . \end{aligned} \tag{75}$$

Theorem 1 is used, and the expression for $S(\omega)$ thus obtained written in the form shown in fig. 12. Here an empty box represents a sum of all diagrams, which satisfy the rules of appendix A, and a box denoted by an 'r' a sum of all diagrams, which satisfy in addition the rule of theorem 2. It is easily verified that all possibilities are exhausted by the four terms in fig. 12. Now we take the residuum for $\omega = E_n$. From eq. (75) this is equal to

$$-\langle 0|G|n\rangle\langle n|B|0\rangle . \tag{76}$$

In fig. 12 only the third term has a pole for $\omega = \underline{E}_n$ originating in the left hand part of the diagram, which from theorem 1 and the factorization theorem[12], is equivalent to a vertex with the frequency ω and the field

$$\frac{1}{4} \sum_{\iota\kappa\lambda\mu} v_{\iota\kappa\lambda\mu} b_\iota b_\kappa \langle 0 | b_\lambda b_\mu \frac{1}{H-\omega} B | 0 \rangle \qquad (77)$$

This expression has from the definition, eq. (51), the residuum

$$- F_n^+ \langle n | B | 0 \rangle . \qquad (78)$$

Dividing by $\langle n | B | 0 \rangle$ we obtain the desired result. q.e.d.

Theorem 3. The matrix element $\langle m | G | n \rangle$ between two excited states is given by the expression in fig. 11c, where

$$G^{(\omega)} \equiv A_0 \frac{1}{H-\underline{\omega}_1-\omega} A_1 \ldots A_{N-1} \frac{1}{H-\omega_N-\omega} A_N , \qquad (79)$$

and the sum, indicated by the box, is restricted as in theorem 2.

Proof. This goes as the proof of theorem 2, taking the residuum of the function

$$\begin{aligned} S(\omega) \equiv \langle 0 | \Big(& B \frac{1}{H-\omega} A_0 \frac{1}{H-\omega_1} A_1 \ldots A_{N-1} \frac{1}{H-\omega_N} A_N \\ & + A_0 \frac{1}{H-\omega_1+\omega} B \frac{1}{H-\omega_1} A_1 \ldots A_{N-1} \frac{1}{H-\omega_N} A_N \\ & + \ldots + A_0 \frac{1}{H-\omega_1+\omega} A_1 \ldots A_{N-1} \frac{1}{H-\omega_N+\omega} A_N \frac{1}{H-E_n+\omega} B \Big) | n \rangle \end{aligned} \qquad (80)$$

for $\omega = \underline{E}_m$. Diagrams, where $\underline{F}_m$ and $\underline{F}_n^+$ are mutually

connected, but disconnected from the vertices $\underline{A}_0, \ldots, \underline{A}_N$ are excluded by the factorization theorem in connection with the free time positions of $\underline{F}_m$ and $\underline{F}_n^+$.

Appendix C

(RPA equations for the oscillating field.)

From theorem 2 (appendix B) the oscillating field $\underline{F}_n$ satisfies the equation in fig. 13a, where the box denoted by an 'r' represents the restricted sum of diagrams specified in the theorem. This equation determines the energy $\underline{E}_n$ and the field $\underline{F}_n$ within a normalization. We understand in this appendix everywhere an arbitrary time order of the external vertices in all diagrams considered.

The normalization is given by the equation in fig. 13c. This is derived from the identity in fig. 13b, where the empty box represents a sum of all diagrams, which satisfy the rules of appendix A. With external vertices A and B the empty box is thus equal (theorem 1 and the arbitrary time order) to

$$\langle 0 | A \frac{1}{H-\omega} B + B \frac{1}{H+\omega} A | 0 \rangle . \qquad (81)$$

Taking the residuum at $\omega = \underline{E}_n$ and dividing by $|\langle 0 | \underline{F}_n | n \rangle|^2$ we obtain fig. 13c.

The equation in fig. 13a can be written as shown in fig. 14a, where the first term on the right hand side is the bubble diagram corresponding to the model space of two-quasiparticle excitations considered, and the second term contains all other contributions to the sum 'r'. Iterating the equation in

fig. 14a we obtain fig. 14b. Here the double dashed line is the effective interaction of the RPA, defined in fig. 14c.

Taking the derivative of fig. 14c with respect to ω we get fig. 14d. Using the eqs. in fig. 14b and 13c we obtain fig. 14e.

In the RPA the effective interaction is assumed frequency independent. Then the second term on the left hand side of fig. 14e vanishes.

An effective interaction with separable terms is written symbolically as

$$H_{eff} = -\frac{1}{2}\sum_{p} \chi_p Q_p^{\dagger} Q_p \,. \tag{82}$$

This expression is understood to have the following meaning: For each term the fields $Q_p^{\dagger}$ and Q_p' act respectively as the left and right hand field of the double dashed line (cf. eq. (71)), and the line itself contributes with the factor χ_p to the value of the diagram, in which it appears. Thus the figs. 14b and 14e are equivalent with eqs. (53) and (54).

Appendix D

(Theorem on loop diagrams. Derivatives with respect to a variation of H_0.)

From theorem 1 (appendix B) and the factorization theorem[12] a sum S of loop diagrams like those in fig. 1 with an arbitrary time order of the external vertices $(A_1,\omega_1),\ldots,(A_N,\omega_N)$

can be written

$$S = \Big\langle \sum_{P} A_{P(1)} \frac{1}{H_0+\omega_{P(1)}} A_{P(2)} \frac{1}{H_0+\omega_{P(1)}+\omega_{P(2)}} A_{P(3)} \cdots A_{P(N-2)} \frac{1}{H_0-\omega_{P(N-1)}-\omega_{P(N)}} A_{P(N-1)} \frac{1}{H_0-\omega_{P(N)}} A_{P(N)} \Big\rangle , \tag{83}$$

where P is an arbitrary permutation of $1,\ldots,N$, and $\langle\,\rangle$ denotes the expectation value in the quasiparticle vacuum. From the definition, eq. (60), eq. (83) can also be written as

$$S = \Big\langle \sum_{P} A_{P(1)} \overbrace{A_{P(2)} \cdots A_{P(N-1)} A_{P(N)}} \Big\rangle , \tag{84}$$

where the sum of frequencies is assigned to a product of operators. We shall prove

$$S = \Big\langle \sum_{P'} [A_{P'(1)}, \overbrace{[A_{P'(2)}, \ldots, [A_{P'(N-1)}, A_N]\ldots]}] \Big\rangle , \tag{85}$$

where P' is an arbitrary permutation of $1,\ldots,(N-1)$.

Proof. Note first that from the algebraic identity

$$\frac{1}{xy} = \frac{1}{x+y}\left(\frac{1}{x}+\frac{1}{y}\right) \tag{86}$$

we get

$$\widehat{XY} = \overbrace{\hat{X}Y + X\hat{Y}} \tag{87}$$

for arbitrary operators X and Y. This implies especially

$$\langle (\hat{X}Y + X\hat{Y}) \rangle = \langle [H_0, \hat{X}\hat{Y}] \rangle = 0. \tag{88}$$

Let us consider the case $N = 4$, which will bring us through all steps of the general proof. From eq. (88) we get

$$< [A_1, \overbrace{[A_2, \overbrace{[A_3, \widehat{A}_4]}]}] >$$

$$= < (A_1 \overbrace{[A_2, \overbrace{[A_3, \widehat{A}_4]}]} + A_2 \overbrace{[A_3, \widehat{A}_4]}\, \widehat{A}_1 \qquad (89)$$

$$+ A_3 \widehat{A}_4 \overbrace{A_2 \widehat{A}_1} + A_4 \overbrace{A_3 \widehat{A}_2}\, \widehat{A}_1) >.$$

We name the four operators in the latter expression $X_1, X_2, X_3,$ and X_4, and understand in the following a summation over the permutations P' so that we can interchange arbitrarily the fields A_1, A_2, and A_3. Thus we have, using eq. (87),

$$X_1 + X_2 = X_1 + A_1 \overbrace{[A_3, \widehat{A}_4] A_2 + [A_3, \widehat{A}_4] \widehat{A}_2}$$

$$= A_1 \overbrace{A_2 [A_3, \widehat{A}_4]} + A_1 \overbrace{[A_3, \widehat{A}_4] \widehat{A}_2} \equiv X_5 + X_2', \qquad (90a)$$

$$X_2' + X_3 = X_2' + A_1 \overbrace{\widehat{A}_4 A_3 \widehat{A}_2 + A_4 A_3 \widehat{A}_2}$$

$$A_1 \overbrace{A_3 \widehat{A}_4 \widehat{A}_2} + A_1 \overbrace{A_4 A_3 \widehat{A}_2} \equiv X_6 + X_7, \qquad (90b)$$

$$X_5 + X_6 = X_5 + A_1 \overbrace{A_2 \widehat{A}_4 A_3 + A_4 \widehat{A}_3}$$

$$= A_1 \overbrace{A_2 A_3 \widehat{A}_4} + A_1 \overbrace{A_2 A_4 A_3} \equiv X_8 + X_9. \qquad (90c)$$

Summing up we get

$$X_1 + X_2 + X_3 + X_4 = X_4 + X_7 + X_9 + X_8 . \qquad (91)$$

With the summation over P' the expectation value of the right hand side is seen to equal just the expression (84). q.e.d.

We note that

$$< [A, \widehat{B}] > = G(A^+, B, \omega), \qquad (92)$$

where ω is the frequency of the vertex B.

In sect. 6 we consider derivatives of loop diagrams with respect to a variation of H_0. Let such a variation be written

$$H_0 \to H_0 - \delta H_0 . \qquad (93)$$

As

$$\delta > = \frac{1}{H_0} \left(\delta H_0 - < \delta H_0 > \right) >, \qquad (94)$$

and

$$\delta \frac{1}{H_0 - \omega} = \frac{1}{H_0 - \omega} \left(\delta H_0 - < \delta H_0 > \right) \frac{1}{H_0 - \omega} \qquad (95)$$

(taking into account the variation of the vacuum energy), it is obvious that the variation of the expression (83) is obtained by inserting simply the field δH_0 with the frequency zero into the particle lines of the loop in all possible

ways, with all possible time orders and making a summation.

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Table 1

Matrix elements of the inertial tensor.

Nucleus	$\left(\frac{1}{2\mathcal{I}_{eff}}\right)_{emp}$ (keV)	$\frac{1}{2\mathcal{I}}$ (keV)	$\langle 0\|\delta\mathcal{I}_{+-}\|\beta\rangle$ (MeV^{-1})	$\langle 0\|\delta\mathcal{I}_{--}\|\gamma\rangle$ (MeV^{-1})	$\langle\beta\|\delta\mathcal{I}_{+-}\|\beta\rangle$ (MeV^{-1})	$\langle\beta\|\delta\mathcal{I}_{--}\|\gamma\rangle$ (MeV^{-1})	$\langle\gamma\|\delta\mathcal{I}_{+-}\|\gamma\rangle$ (MeV^{-1})
^{152}Sm	21.2	17.8	19.8	14.3	4.6	1.0	-1.8
^{154}Gd	21.3	18.3	17.9	13.5	- 8.1	3.6	-2.3
^{156}Gd	15.0	14.2	6.5	15.2	16.0	2.4	-1.8
^{158}Gd	13.5	12.4	18.4	15.0	31.8	-2.8	0.0
^{160}Dy	14.6	13.5	15.2	15.3	10.5	1.3	0.8
^{166}Er	15.3	13.6[a]	21.6[a]	13.7	24.8[a]	0.7[a]	5.3
^{170}Yb	14.0	13.1	17.1	11.2	4.9	-3.3	3.0
^{174}Hf	15.3	13.4	25.2	12.4	12.3	-1.2	-2.0
^{176}Hf	14.7	12.7	29.3	11.4	14.3	0.4	-0.1
^{178}Hf	15.6	14.5	15.4	10.9	3.9	0.1	2.6
^{182}W	16.8	15.6	13.5	11.7	10.7	1.4	4.7
^{184}W	18.6	17.2	12.0	12.1	-22.2	-0.4	4.7
^{186}W	20.8	18.6	14.1	11.9	-19.6	-1.4	3.7
^{186}Os	23.3	20.1	16.8[a]	10.6	5.5[a]	3.6[a]	6.3
^{188}Os	26.7	22.7	14.4[a]	11.7	- 3.9[a]	0.0[a]	2.5

[a] Estimated energy E_β.

Table 2

Matrix elements of $\delta\mathcal{I}_{\alpha\beta}$ in the adiabatic approximation [a].

Nucleus	$\langle 0\|\delta\mathcal{I}_{+-}\|\beta\rangle$ (MeV^{-1})	$\langle 0\|\delta\mathcal{I}_{--}\|\gamma\rangle$ (MeV^{-1})	$\langle\beta\|\delta\mathcal{I}_{+-}\|\beta\rangle$ (MeV^{-1})	$\langle\beta\|\delta\mathcal{I}_{--}\|\gamma\rangle$ (MeV^{-1})	$\langle\gamma\|\delta\mathcal{I}_{+-}\|\gamma\rangle$ (MeV^{-1})
^{152}Sm	23.7	12.4	10.9	0.4	-4.0
	19.8	14.3	4.6	1.0	-1.8
^{166}Er	24.3[b]	12.9	8.9[b]	0.9[b]	1.7
	21.6[b]	13.7	24.8[b]	0.7[b]	5.3
^{186}W	27.7	11.3	22.0	4.7	-0.1
	14.1	11.9	-19.6	-1.4	3.7

[a] Upper value. Lower value as in table 1.

[b] Estimated energy E_β.

Table 3

The quantities g_R and $g_K - g_R$. Matrix elements of $g^{(1)}_{\alpha\beta}$.

Nucleus	g_R	$g_K - g_R$	$\langle 0 \mid g^{(1)}_{+} \mid \beta \rangle$ (10^{-2})	$\langle 0 \mid g^{(1)}_{-} \mid \gamma \rangle$ (10^{-2})	$\langle \beta \mid g^{(1)}_{+} \mid \beta \rangle$ (10^{-2})	$\langle \beta \mid g^{(1)}_{-} \mid \gamma \rangle$ (10^{-2})	$\langle \gamma \mid g^{(1)}_{+} \mid \gamma \rangle$ (10^{-2})
^{152}Sm	0.34	-0.10	2.4	-0.4	-0.6	-0.9	-1.1
^{154}Gd	0.42	-0.12	-0.5	-1.0	5.1	-2.2	1.4
^{156}Gd	0.40	-0.08	6.7	-2.0	-6.6	-1.3	1.2
^{158}Gd	0.34	0.02	4.6	-0.7	-0.3	1.0	0.2
^{160}Dy	0.37	0.13	1.8	0.4	-4.4	1.0	2.2
^{166}Er	0.30	0.07	-4.6[a]	0.8	-12.0[a]	-0.5[a]	-0.4
^{170}Yb	0.34	0.07	1.3	0.0	6.0	0.8	-0.2
^{174}Hf	0.24	-0.03	2.7	1.2	6.5	1.6	0.0
^{176}Hf	0.26	-0.03	-3.3	0.6	-1.7	-0.1	-0.5
^{178}Hf	0.24	0.12	-3.3	-0.3	17.8	0.0	0.0
^{182}W	0.26	-0.12	0.4	-1.3	15.7	-0.9	-2.3
^{184}W	0.29	-0.20	-0.6	-2.4	-5.0	-0.3	-3.0
^{186}W	0.31	-0.16	-1.4	-1.6	-5.3	0.0	-3.6
^{186}Os	0.31	-0.16	-1.7[a]	-1.9	5.7[a]	-1.7[a]	3.4
^{188}Os	0.28	0.04	-0.8[a]	0.9	6.5[a]	0.7[a]	-0.8

[a] Estimated energy E_R.

Table 4

Calculated B(M1) values [a].

Nucleus	$B(M1, 2_\beta \to 2_0)$		$B(M1, 2_\gamma \to 2_0)$		$B(M1, 2_\beta \to 2_\gamma)$
	calc.	emp.	calc.	emp.	calc.
^{152}Sm	[h](+) 8.2	(+) 0.3 ± 0.1 [b]	(-) 0.0	(-) 3.3 ± 0.2 [b]	(-) 0.4
^{154}Gd	(-) 0.7	(+) 1.0 $^{+0.9}_{-0.3}$ [b]	(-) 1.1	(-) 4.0 $^{+0.9}_{-0.7}$ [b]	(-) 0.2
^{156}Gd	(+)59.5	(-) 2.4 $^{+1.8}_{-1.3}$ [b]	(-) 1.6	(-) 0.9 $^{+0.4}_{-0.3}$ [b]	(-) 3.4
^{158}Gd	(+)18.5		(-) 0.4		(+) 0.0
^{160}Dy	(+) 5.3		(+) 0.0	(-) 2.3 $^{+1.0}_{-0.6}$ [b]	(+) 0.4
^{166}Er	(-) 6.0[g]		(+) 0.7[g]	(-) 0.5 $^{+1.3}_{-0.5}$ [b]	(+) 0.2[g]
^{170}Yb	(+) 0.5[f]		(-) 0.0[f]		(+) 0.3[f]
^{174}Hf	(+) 2.8	(-) < 5.4 [c]	(+) 0.5		(+) 0.5
^{176}Hf	(-)10.0	< 5.8 [d]	(+) 0.0		(+) 0.2
^{178}Hf	(-)34.9	(+) < 2.7 [i]	(-) 0.1	(-) 0.4 $^{+2.3}_{-0.4}$ [b]	(+) 0.2
^{182}W	(-) 1.8	(-) 0.6 $^{+1.0}_{-0.4}$ [e]	(-) 0.3	(+) 0.9 $^{+0.8}_{-0.6}$ [b]	(-)26.8
^{184}W	(-) 0.3	(+) 8 $^{+7}_{-3}$ [b]	(-) 3.2	(-) 0.8 ± 0.1 [b]	(+) 0.0
^{186}W	(-) 2.3	(+) 0.1 $^{+0.3}_{-0.1}$ [b]	(-) 1.7	(-) 1.5 $^{+1.3}_{-0.7}$ [b]	(+) 0.0
^{186}Os	(-) 6.8 [g]		(-) 1.3 [g]	(-) 2.2 $^{+2.5}_{-1.8}$ [b]	(-) 1.1[g]
^{188}Os	(-) 3.5 [g]		(+) 0.9 [g]	(-) 1.6 $^{+1.1}_{-0.5}$ [b]	(+) 0.1[g]

Table 4 (continued)

[a] Units of $10^{-4}(\frac{e}{2Mc})^2$

[b] Calculated from $B(M1, 2_\mu \to 2_0) = \frac{0.14}{1+\delta^2}(E_{2_\mu} - E_{2_0})^2 \frac{B(E2, 2_\mu \to 2_0)_{eff}}{B(E2, 2_\mu \to 0_0)} B(E2, 0_0 \to 2_\mu)$.
δ-values from the compilation in ref. [43]. Branching ratios from refs. [35, 44, 45].
$B(E2, 0_0 \to 2_\mu)$ as in table 5.

[c] Ref. [31].

[d] Ref. [33].

[e] Ref. [44].

[f] Mixing between the β- and γ-band not included.

[g] Estimated energies in the β-band.

[h] The sign of $\langle 2_f \| M(M1) \| 2_i \rangle$ is given in parentheses.

[i] Ref. [45].

Table 5

Calculated values of $B(E2, 0_o \rightarrow 2_\mu)$ [a].

Nucleus	$B(E2, 0_o \rightarrow 2_\beta)$ cal.	exp.	$B(E2, 0_o \rightarrow 2_\gamma)$ cal.	exp.
^{152}Sm	0.008	0.023[c]	0.15	0.08[h]
^{154}Gd	0.004	0.024[c]	0.17	0.17[h]
^{156}Gd	0.013	0.010[c]	0.13	0.12[h]
^{158}Gd	0.010		0.13	
^{160}Dy	0.015		0.17	0.14[h]
^{166}Er	-		0.17[b]	0.15[h]
^{170}Yb	0.006		0.15	0.09[h]
^{174}Hf	0.056	0.062[d]	0.14	
^{176}Hf	0.021	0.025[e]	0.13	
^{178}Hf	0.034	<0.002[f]	0.11	
^{182}W	0.059	0.028[g]	0.06	0.12[g]
^{184}W	0.039	0.021[g]	0.08	0.13[g]
^{186}W	0.021	0.009[g]	0.12	0.15[g]
^{186}Os	-		0.19[b]	0.19[g]
^{188}Os	-		0.26[b]	0.25[g]

cont.

Table 5 (continued)

[a] Units of $(e\ b)^2$.

[b] Estimated energies in the β-band.

[c] Ref. 2).

[d] Ref. 31).

[e] Ref. 33).

[f] Ref. 45).

[g] Ref. 44).

[h] Compilation by G.B. Hagemann 46).

[i] The mixing of the β- and the γ-band was not included.

Figure Captions

Fig. 1 a) Matrix element of $\delta\mathcal{J}_{\alpha\beta}$ between the ground state and an excited state.

b) Matrix element between two excited states.

Fig. 2 Graphical representation of the cranking formula.

Fig. 3 Quadrupole coupling constants obtained by a fit to the empirical energies. Crosses: β-vibration. Circles: γ-vibration. Dashed lines: The expression (68) with $p = 50$ MeV fm^{-4}, $\beta = 0.3$.

Fig. 4 The matrix element $\langle 0|\underline{h}_0|\beta\rangle$. Explanation in the text.

Fig. 5 The matrix element $\langle 0|\underline{h}_{-2}|\gamma\rangle$. Explanation in the text.

Fig. 6 The matrix element $\langle \beta|\underline{h}_{-2}|\gamma\rangle$. Explanation in the text.

Fig. 7 The quantity $\langle\beta|\underline{h}_0|\beta\rangle - \langle 0|\underline{h}_0|0\rangle$. Crosses: experiment. Circles: Calculation of ref.[7]. Further explanation in the text.

Fig. 8 The quantity $\langle\gamma|\underline{h}_0|\gamma\rangle - \langle 0|\underline{h}_0|0\rangle$. Explanation in the text.

Fig. 9 Plot of $(\underline{E}_{\underline{I}} - \underline{E}_{\underline{I}-1})/(2\underline{I})$ vs. $\underline{I}^2 - 4$ for four γ-bands. The arrow on the ordinate axis indicates $(1/(2\mathcal{J}_{eff}))_{emp}$.

Fig. 10 Example of a diagram.

Fig. 11 Illustrations to the theorems 1-3.

Fig. 12 Illustation to the proof of theorem 2.

Fig. 13 Illustration to appendix C.

Fig. 14 Illustration to appendix C.

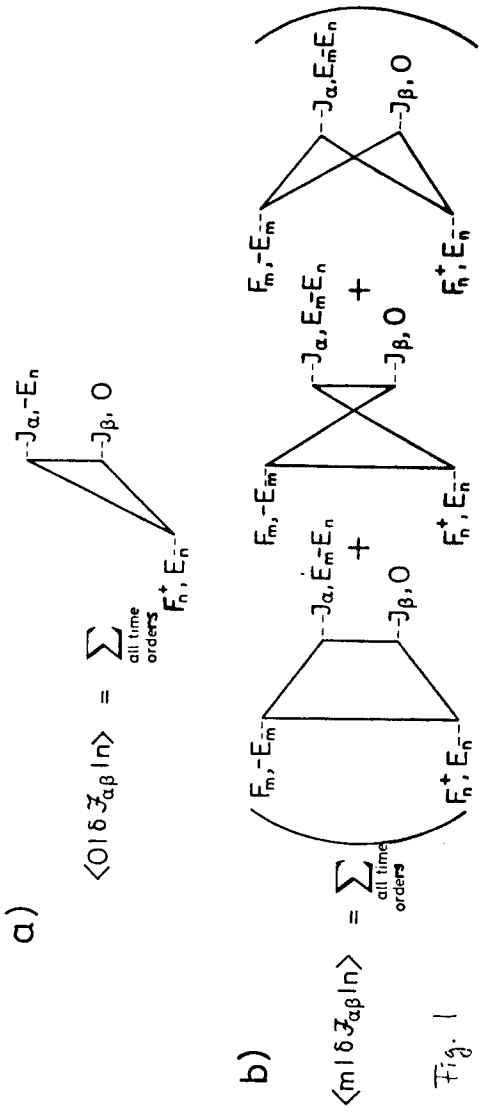


Fig. 1

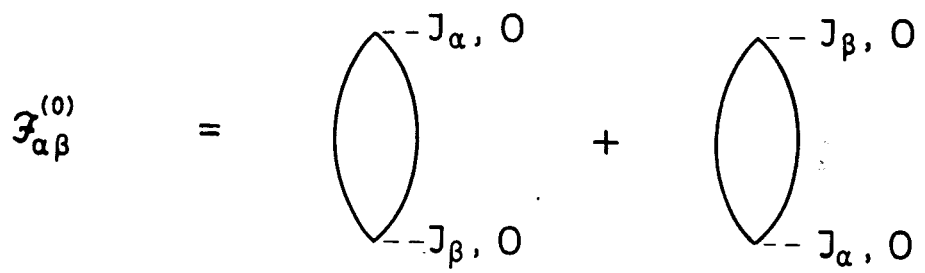

$\mathcal{F}^{(0)}_{\alpha\beta}$ =
$J_\alpha, 0$
$J_\beta, 0$
+
$J_\beta, 0$
$J_\alpha, 0$

Fig. 2

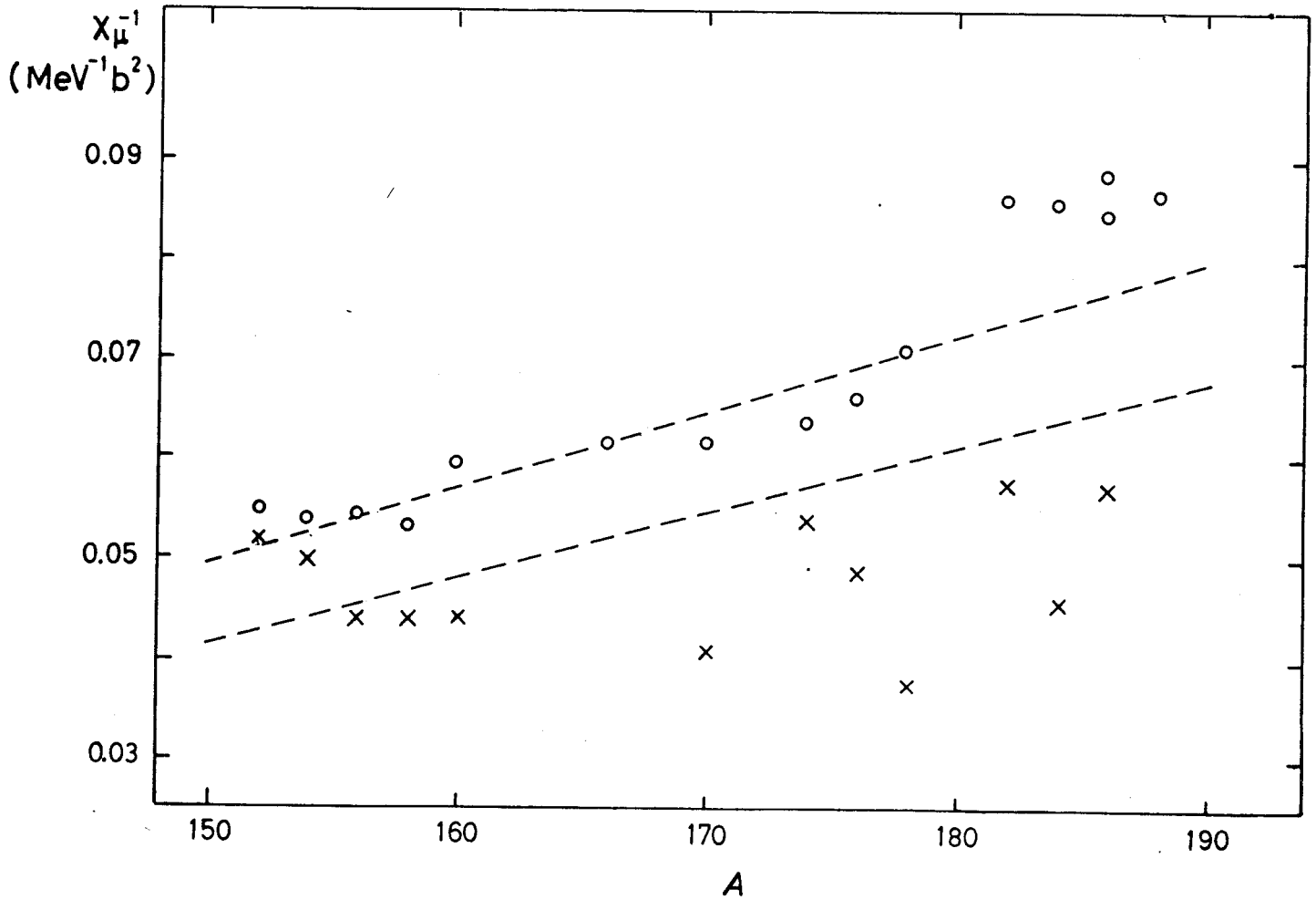

χ_μ^{-1}
(MeV^{-1}b^2)
0.09
0.07
0.05
0.03
150
160
170
180
190
A

Fig. 3

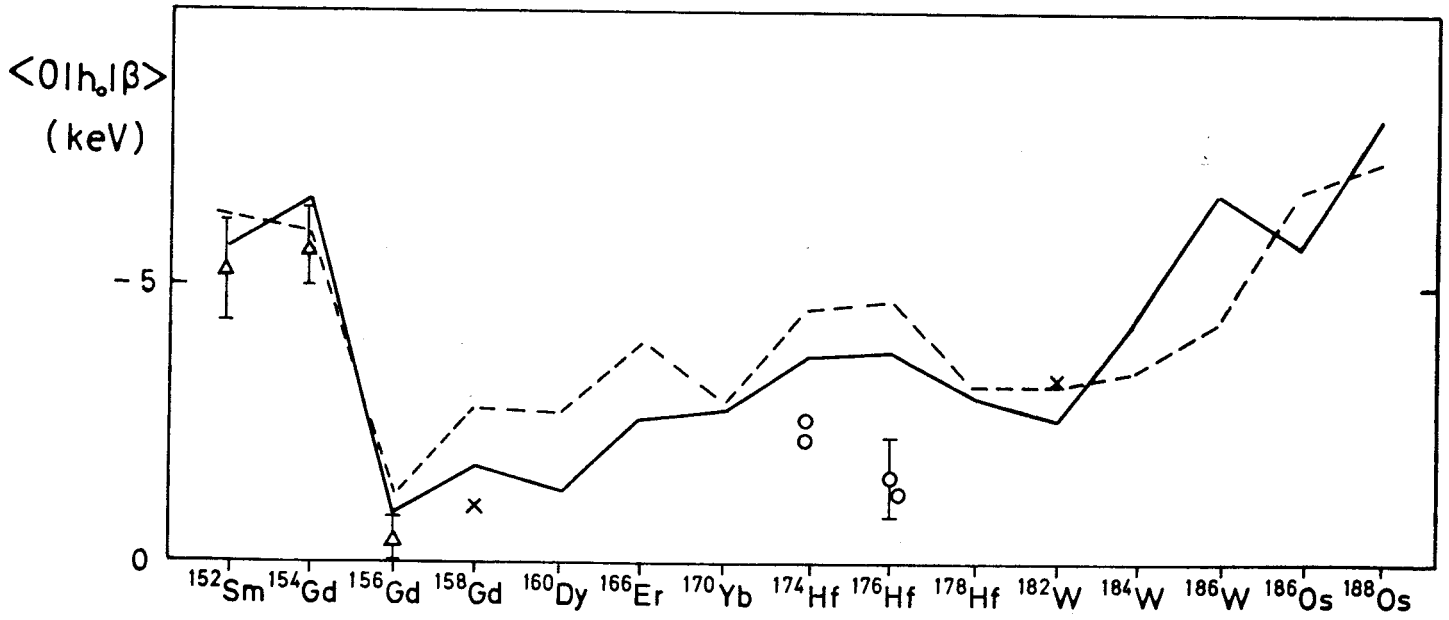

<0|h0|β>
(keV)
-5
0
152Sm 154Gd 156Gd 158Gd 160Dy 166Er 170Yb 174Hf 176Hf 178Hf 182W 184W 186W 186Os 188Os

Fig 4

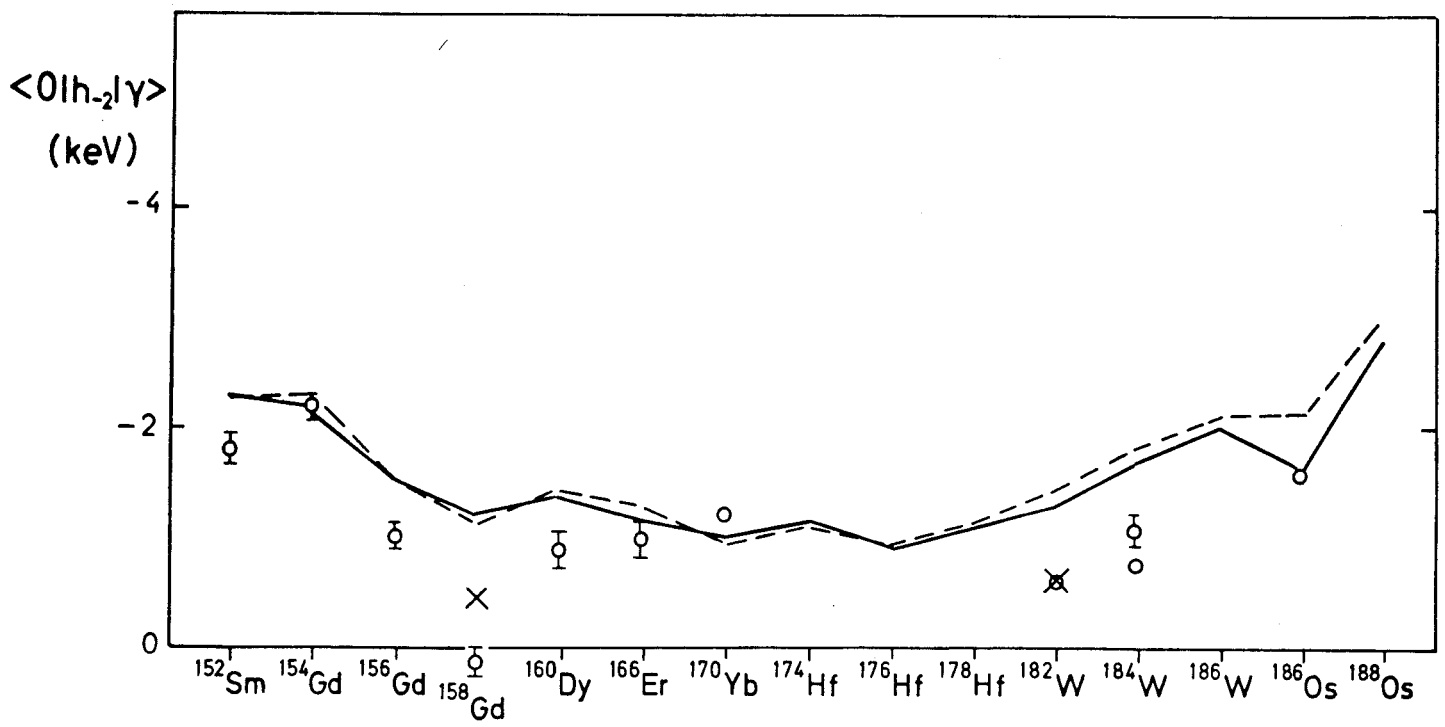

<0|h-2|γ>
(keV)
-4
-2
0
152Sm 154Gd 156Gd 158Gd 160Dy 166Er 170Yb 174Hf 176Hf 178Hf 182W 184W 186W 186Os 188Os

Fig. 5

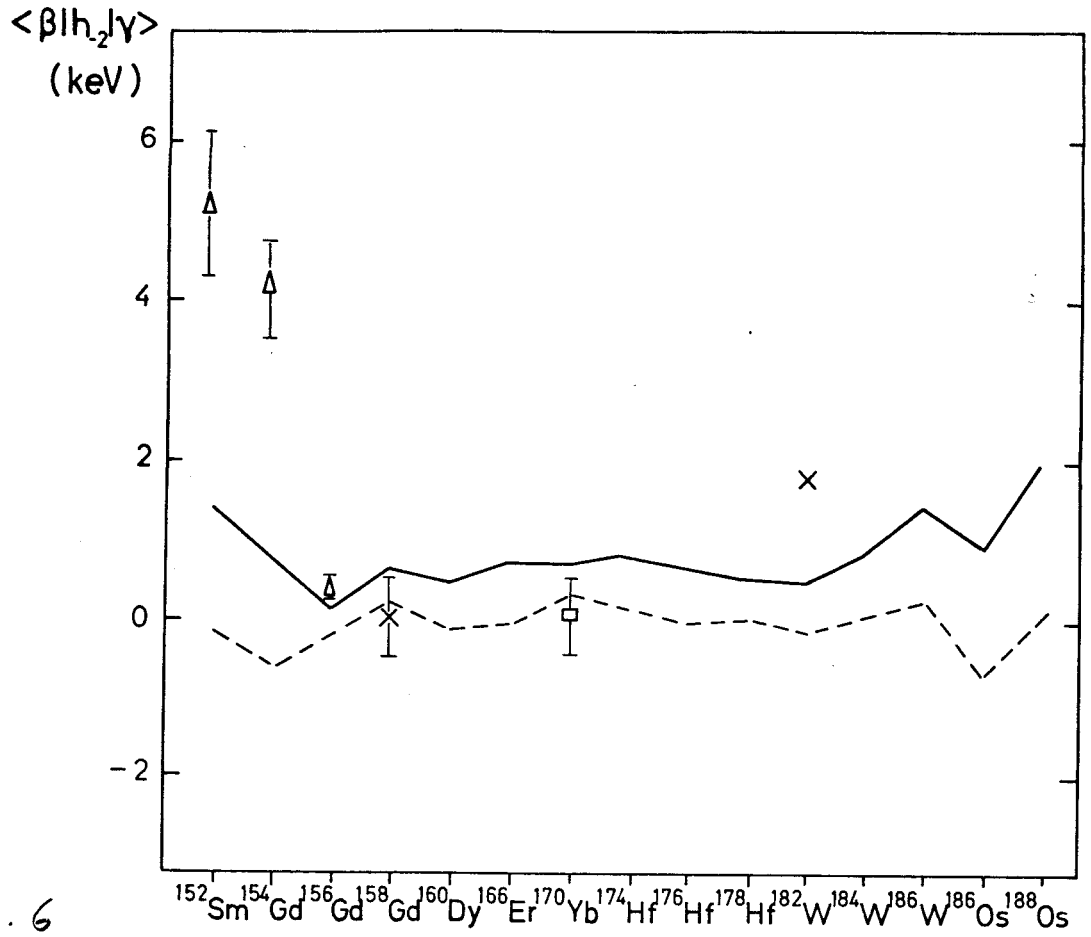

<β|h2|γ>
(keV)
6
4
2
0
−2
152Sm 154Gd 156Gd 158Gd 160Dy 166Er 170Yb 174Hf 176Hf 178Hf 182W 184W 186W 186Os 188Os

Fig. 6

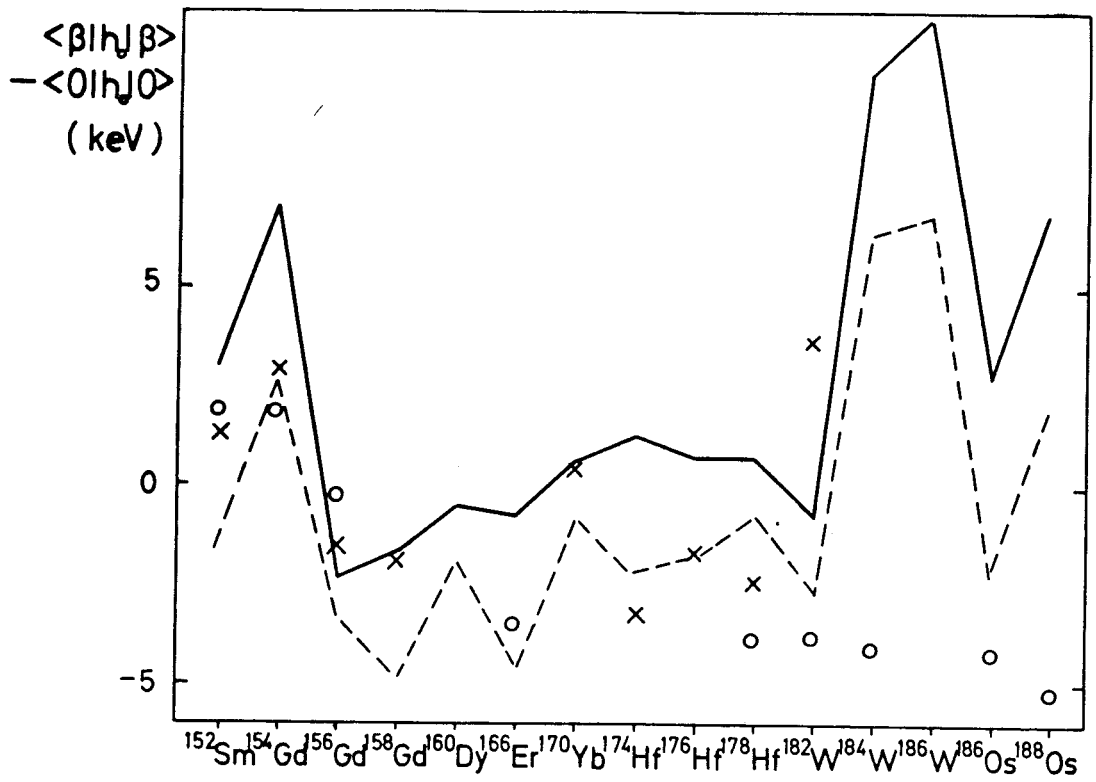

<β|hJ|β>
−<0|hJ|0>
(keV)
5
0
−5
152Sm 154Gd 156Gd 158Gd 160Dy 166Er 170Yb 174Hf 176Hf 178Hf 182W 184W 186W 186Os 188Os

Fig. 7

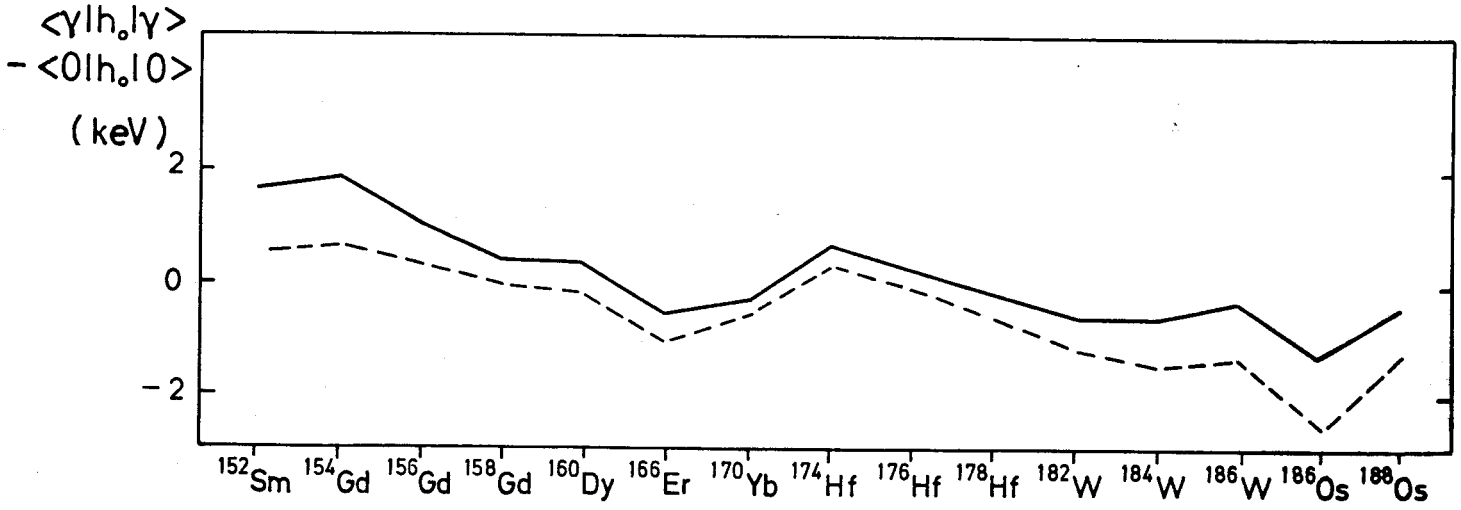

<γ|h₀|γ>
-<0|h₀|0>
(keV)
2
0
-2
152Sm 154Gd 156Gd 158Gd 160Dy 166Er 170Yb 174Hf 176Hf 178Hf 182W 184W 186W 186Os 188Os

Fig. 8

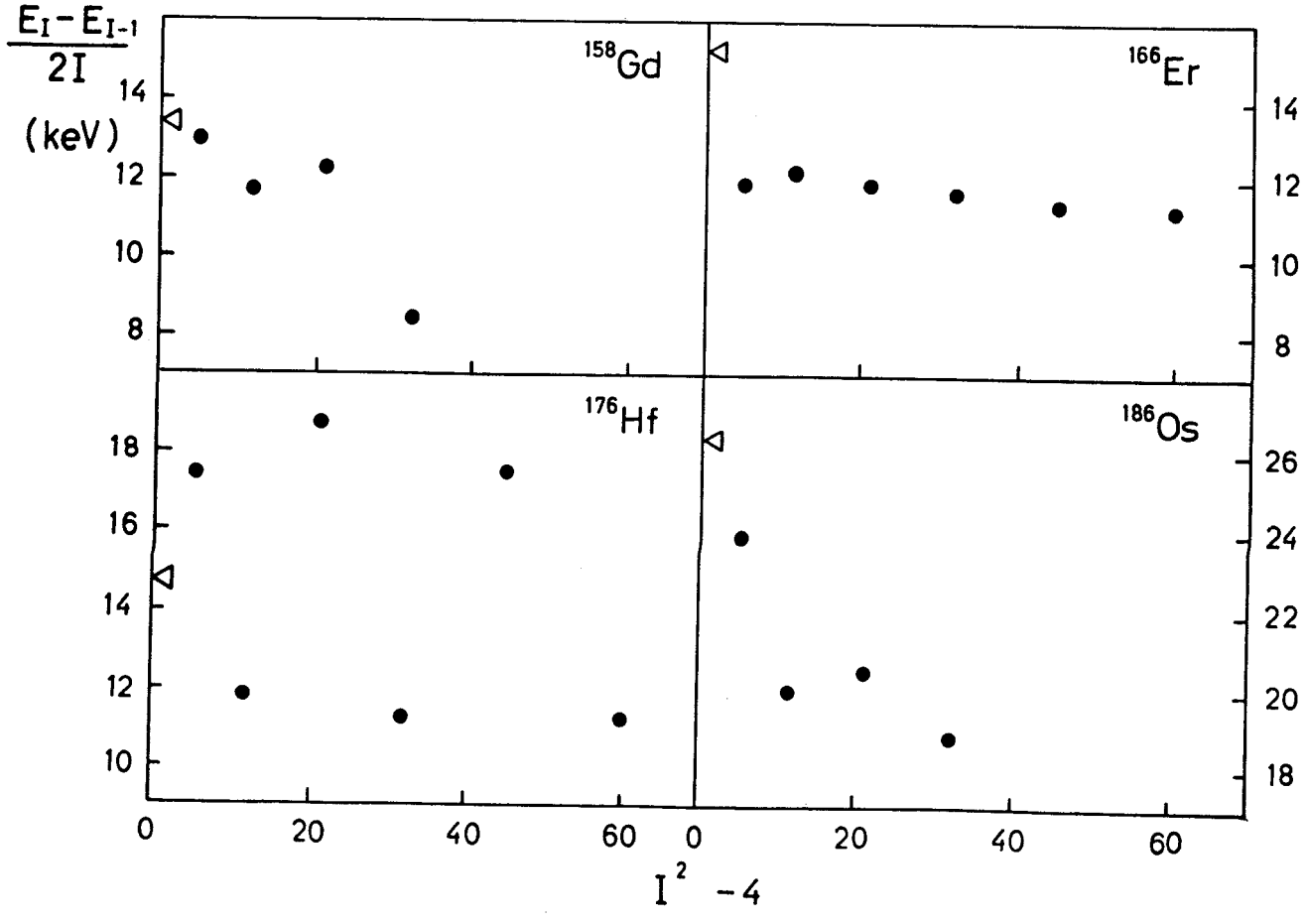

E_I - E_I-1
2I
(keV)
158Gd
166Er
176Hf
186Os
I² -4

Fig. 9

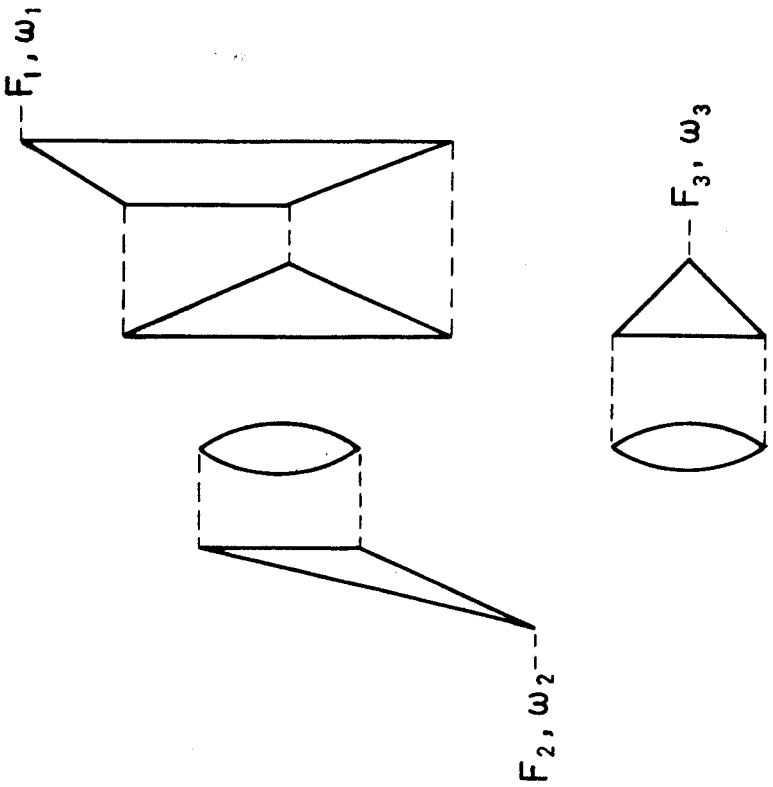
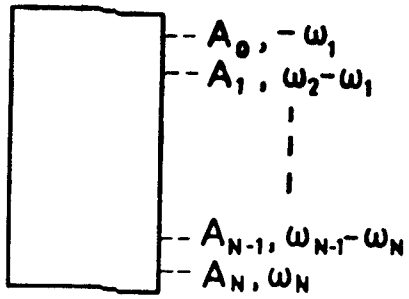


Fig. 10

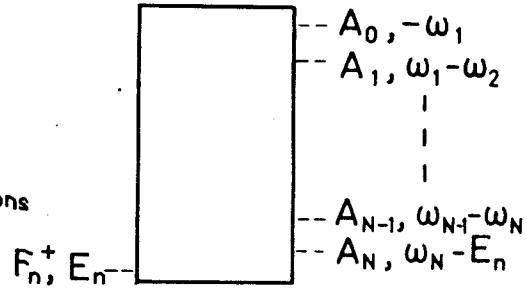
a)

$\langle 0 \mid G \mid 0 \rangle$ =



b)

$\langle 0 \mid G \mid n \rangle$ = $\sum$ time positions of F_n^+



c)

$\langle m \mid G \mid n \rangle$ = $\langle m \mid n \rangle$ $\langle 0 \mid G^{(-E_n)} \mid 0 \rangle$

+ $\sum$ time positions of F_m and F_n^+

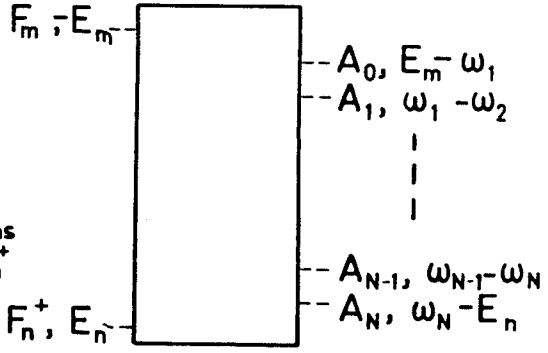


Fig. 11

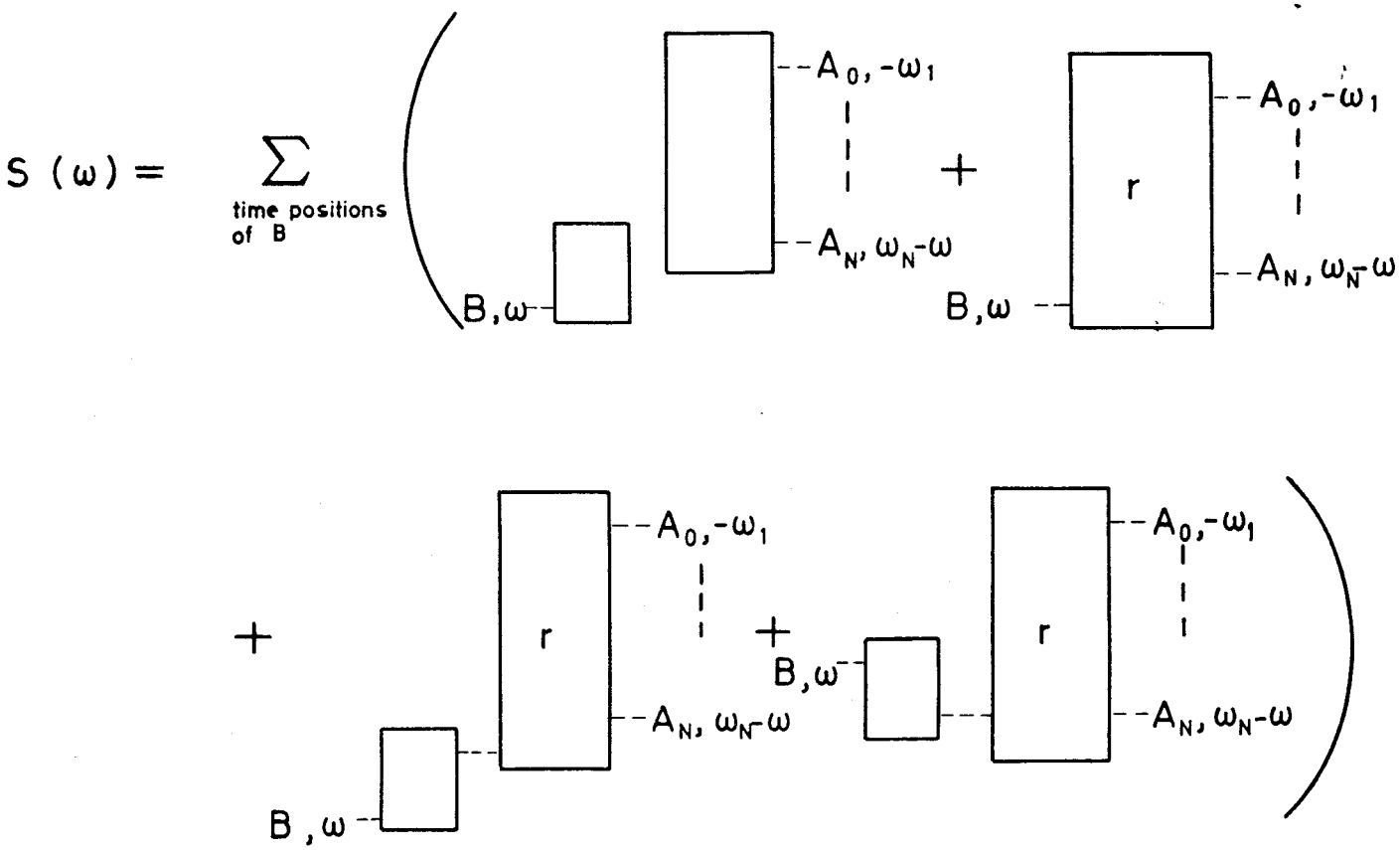

S (ω) =
time positions of B
A0, -ω1
AN, ωN-ω
B, ω
r
+

Fig. 12

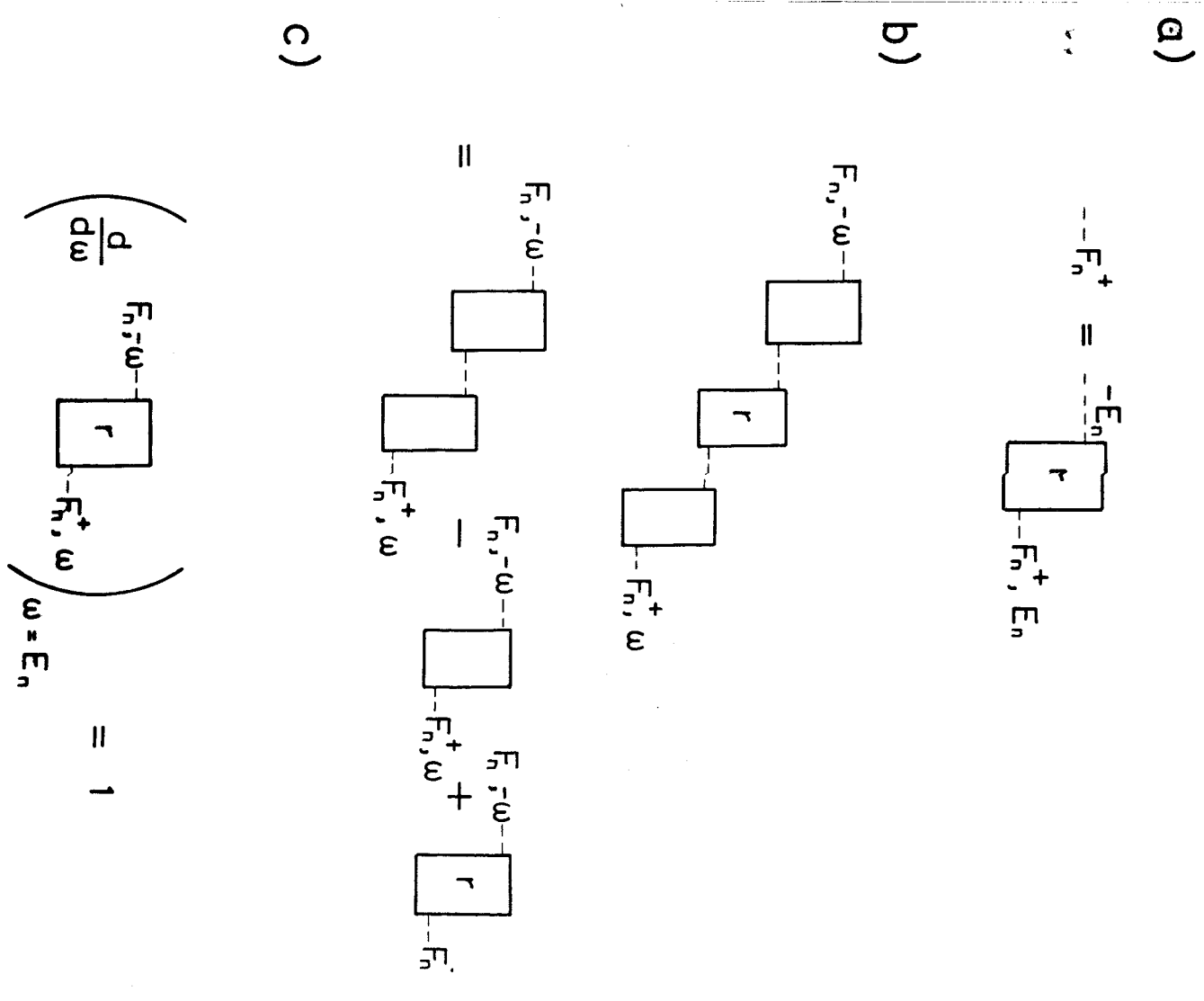

a)
b)
c)
Fn+
Fn, -ω
Fn, ω
Fn, En
r
ε = En
d/dε
1
+

Fig. 13

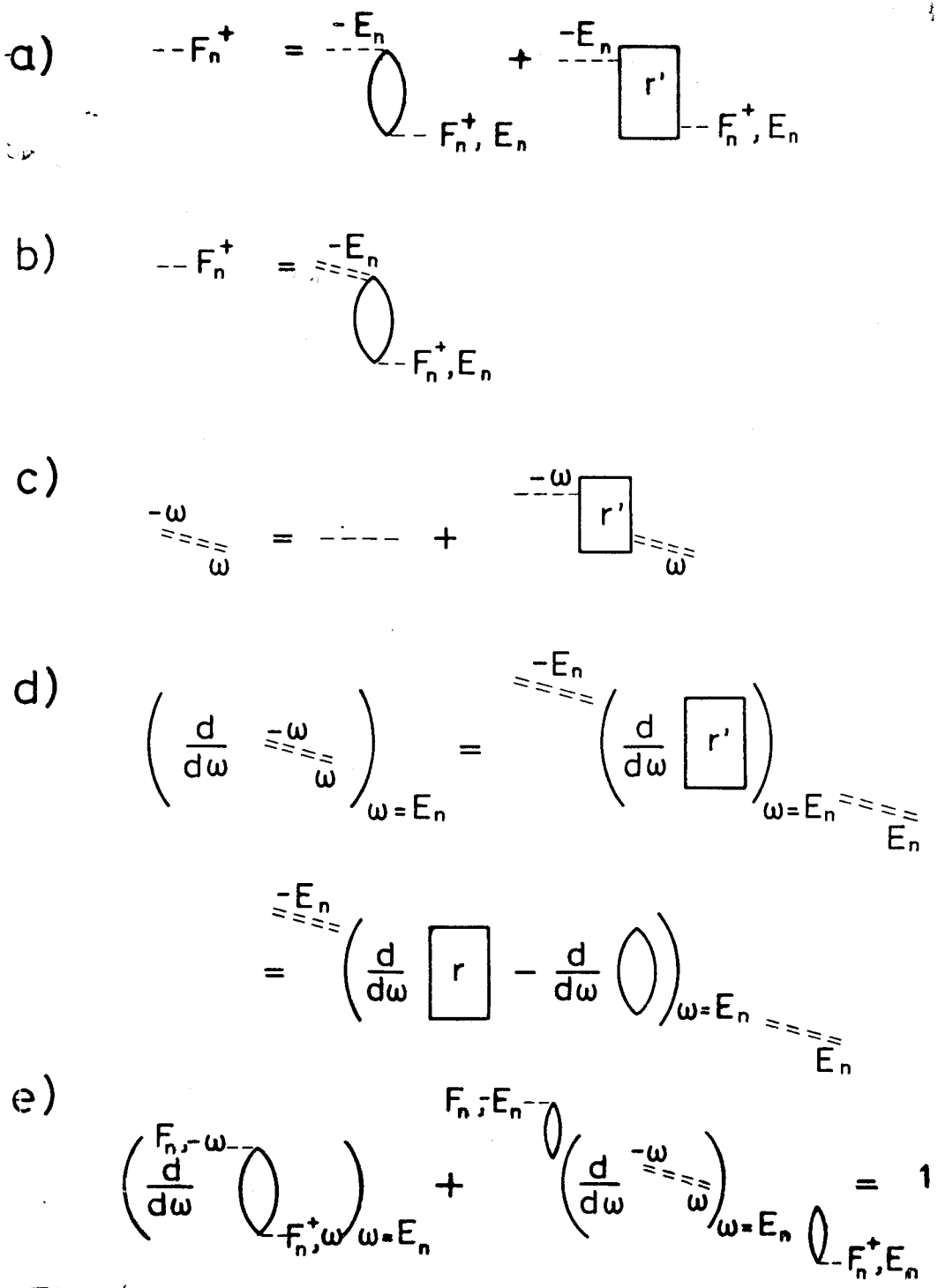

a)
$-F_n^+$ = $-E_n$ F_n^+, E_n + $-E_n$ r' $-F_n^+, E_n$
b)
$-F_n^+$ = $-E_n$ $-F_n^+, E_n$
c)
$-\omega$ ω = + $-\omega$ r' ω
d)
$\left(\frac{d}{d\omega}\right)_{\omega=E_n}$ $-\omega$ ω = $-E_n$ $\left(\frac{d}{d\omega}\right)_{\omega=E_n}$ r' E_n
= $-E_n$ $\left(\frac{d}{d\omega}\right)_{\omega=E_n}$ r − $\frac{d}{d\omega}$ E_n
e)
$F_n, -\omega$ $\left(\frac{d}{d\omega}\right)_{\omega=E_n}$ $-F_n^+, \omega$ + $F_n, -E_n$ $\left(\frac{d}{d\omega}\right)_{\omega=E_n}$ $-\omega$ ω $-F_n^+, E_n$ = 1

Fig. 14